HF radar investigation of source terms in the Hasselmann equation
The spark that ignited my interest

“... the values of different wind input terms scatter by a factor of 300 – 500 % ([1], [2])”

from


which referenced


The spread of models for wind-wave growth:

Fig. 1. Dependence of wind-wave growth rate on non-dimensional frequency $\omega U_{10}/g$ in log- and linear scales given by different experimental parameterizations (see legends).

The Hasselmann equation

The evolution of the wave field is often described by the action balance equation

\[
\frac{\partial N}{\partial t} + \nabla_x \cdot (\nabla_\kappa \Omega N) - \nabla_\kappa \cdot (\nabla_x \Omega N) = S_{in} + S_{nl} + S_{dis} = 0
\]

under steady state conditions

where the wave action density \( N(\kappa) \) is related to the wave displacement spectrum \( S(\kappa) \) by

\[
N(\kappa) = \frac{\rho_w g S(\kappa)}{\sigma(\kappa)}
\]

with \( \sigma \) the intrinsic frequency,

\[
\Omega = \kappa \cdot \vec{U} + \sigma
\]

input from the wind

spectral flux due to nonlinear interactions

loss via dissipative processes

Steady state requires directional bimodality (Komen et al, 1984)
The Hasselmann equation II

For a wide range of conditions, the action balance equation reduces to the familiar form in terms of the energy spectral density - the Hasselmann equation:

$$\frac{\partial S(\kappa)}{\partial t} + \nabla_x \cdot \left( \nabla_\kappa \Omega S(\kappa) \right) = S_{in} + S_{nl} + S_{dis}$$

and it is this that forms the basis of the main wave modelling codes. There it is convenient to use frequency-angle coordinates,

$$\frac{1}{\rho_w g} N(\kappa) d\kappa = \frac{S(\kappa) d\kappa}{\sigma(\kappa)} = \frac{g^3}{2\sigma^4} S(\sigma, \varphi) d\sigma d\varphi$$

The challenge is to find mathematical models for the source terms $S_{in}$, $S_{nl}$ and $S_{dis}$.
Dissipation mechanisms for wind-generated surface gravity waves

\[ \frac{\partial N}{\partial t} + \nabla_x \cdot (\nabla_\kappa \Omega N) - \nabla_\kappa \cdot (\nabla_x \Omega N) = S_{\text{in}} + S_{\text{nl}} + S_{\text{dis}} \]

\[ S_{\text{dis}} = S_{\text{wc}} + S_{\text{mv}} + S_{\text{tv}} + S_{\text{ma}} + S_{\text{wci}} + S_{\text{iw}} + S_{\text{bf}} + S_{\text{biwb}} + S_{\text{wiw}} + S_{\text{ice}} + S_{\text{BF}} \]

- white-capping
- molecular viscosity
- turbulent viscosity
- Marangoni damping
- wave-current interactions
- internal wave coupling
- bottom friction
- bottom-induced wave breaking
- wave-induced winds
- sea ice coupling
- Benjamin Feir instability / Fermi-Pastra-Ulam recurrence
Wavenumber-direction space or frequency-direction space?

Grid design and other considerations favour frequency-direction space for numerical wave modelling (e.g., SWAN, WAVEWATCH III).

From a radar perspective, wavenumber-direction space is better as the scattering integrals have an elegant physical interpretation in terms of multiple Bragg (resonant) scattering.

The Jacobian is not ill-behaved, so both forms have been used by the HF radar community.

There is a nice paper on the transformation by Hsu et al., in *China Ocean Eng.*, Vol. 25, No. 1, pp. 133 – 138, 2011.

The SPM2 mapping from directional wave spectrum to radar Doppler spectrum

\[ \tilde{S} = \int d\kvec \tilde{R}(\kvec', \kvec) \delta(\kvec' - \kvec + 2\kvec \cdot \hat{n} \cdot \hat{n}) + \]
\[ + \int d\kvec_1 F^{(1)}(\kvec', \kvec, \kvec_1) S(\kvec_1) + \]
\[ + \int \int d\kvec_1 d\kvec_2 F^{(2)}(\kvec', \kvec, \kvec_1, \kvec_2) S(\kvec_1) S(\kvec_2) + \]

+ ..... 

NB: incomplete 4th order contributions for non-Gaussian surfaces
Currents, wind and wave information from the ACORN SA Gulf radar system

- Current velocity
- Significant waveheight
- Inferred surface wind
Directional wave spectrum sampling levels

- Any HF radar can deliver (a)
- Multi-frequency radars can also do (b)
- If two radars illuminate the same patch of ocean, they can deliver (c)
- If bistatic mode is enabled, they can do (d)
- If the signal is uncorrupted and inversion is enabled, a single radar can do (e)
How might HF radar contribute to refining the source term models?

What we have to offer is the ability to monitor properties of the directional wave spectrum on kilometre scale resolution, over large areas ($10^4$ – $10^6$ sq km), with a refresh rate of order $100$ – $1000$ s.

We do this by interacting directly with the surface gravity waves, not indirectly via capillary waves.

We have developed libraries of radar signatures of quite a number of ocean and atmospheric phenomena, some validated by experiment, others awaiting opportunities to put them to the test.

We recognise that solving an equation of the form

$$X + Y + Z = Q$$

for the functions $X$, $Y$ and $Z$, given only measurements of function $Q$ sounds implausible if not impossible, but hope springs from several considerations:

- $X$ is relatively well understood, though not easy to evaluate.
- The equation plays out in the arena of **space and time and wavenumber and direction**, which we can sample with resolution comparable with or better than the characteristic spatial and temporal scales of $X$, $Y$ and $Z$.
- There may be particular conditions in which either $Y$ or $Z$ may be assumed to dominate the other, setting aside $X$.
- There may be particular conditions in which either $Y$ or $Z$ may vary much more rapidly than the other, setting aside $X$.
- Perhaps conditions far from equilibrium will stress-test the equation.

**THE BRICKS WITHOUT STRAW CHALLENGE**
Representative omni-directional spectral structure of a non-stationary sea and HF radar Bragg frequencies
Measured directional response of a wave field under a progressive change in wind direction

Anderson, 2012; adapted from Perrie and Toulany, 1995
Modelling examples: veering wind effects over 80 minutes

Evolution of spectrum during veering wind event

15 MHz

Evolution of spectrum during veering wind event - common normalisation

25 MHz
Discrepancy introduced by assuming adiabatic development

Doppler spectrum during early stage of evolution under veering wind

Doppler spectrum during early stage of evolution assuming adiabatic development

Discrepancy
Target observable #15: Tropical convective cells
Sea state variation at a squall line

Wind speed $\sim 20$ m/s gusting to 25 m/s

Roswold, 2010
Short waves excited by convective cells

Convective disturbance in calm conditions 95:342

Convective disturbance occurring as longer waves decay following a brief storm

Schulz, 1995

NB figures have different scales
Modelled HF radar Doppler spectrum evolution during the lifetime of the convective cell computed from measured wave spectra.
What other options do we have to help unravel the Hasselmann equation?

When Nature doesn’t give us the waves we want, **MAKE THEM**!

When Nature’s standard boundary conditions at the sea surface are not conducive to our study, **FIND OTHERS**!

When it’s all too hard for humans, **ASK A COMPUTER**!
Generic form of the Kelvin wake and computed wake patterns for two frigates

Comparison of angular spectra for two frigates, at two speeds

Strong prospect of discrimination capability
Surface wave spectra for the three components of the scattering integral.

(i) Ambient wave spectrum
Established theory and practice, but parametric wave models must be used with care because of environmental nonstationarity.

(ii) Wake spectrum
Hydrodynamics well understood within the framework of potential theory, as implemented under various approximations. Use of RANS and other CFD approaches enables more realistic modelling and description of turbulence.

(iii) Total spectrum
Within the context of weak turbulence theory, no new physics enters but for our application we need to quantify the dissipation rate for waves on the wake manifold under the prevailing ambient wave spectrum.
The calculation of the scattering from the composite wave + wake surface proceeds by assuming the statistical independence of the respective wave height contributions, analogous to the Barrick SPM2 approach, to partition the directional wave spectrum,

\[ S(\vec{k}) = S_{\text{wake}}(\vec{k}) + S_{\text{wave}}(\vec{k}) \]

so we can substitute and expand the integrands in the scattering formula,

\[
\begin{align*}
\vec{D}(\vec{k}_{\text{scat}}, \vec{k}_{\text{inc}}; \omega) &= \int d\vec{k}_1 F_1(\vec{k}_{\text{scat}}, \vec{k}_{\text{inc}}, \vec{k}_1)[S_{\text{wake}}(\vec{k}_1) + S_{\text{wave}}(\vec{k}_1)] \\
&\quad + \int\int d\vec{k}_1 d\vec{k}_2 F_2(\vec{k}_{\text{scat}}, \vec{k}_{\text{inc}}, \vec{k}_1, \vec{k}_2)[S_{\text{wake}}(\vec{k}_1) + S_{\text{wave}}(\vec{k}_1)] \times [S_{\text{wake}}(\vec{k}_2) + S_{\text{wave}}(\vec{k}_2)]
\end{align*}
\]
HF radiowave scattering from a composite wake – wave surface III

\[
\vec{D}(\vec{k}_{scat}, \vec{k}_{inc}; \omega(\kappa)) = \int d\vec{k}_1 F_1(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1)[S_{wave}(\vec{k}_1)]
\]

\[
+ \int d\vec{k}_1 F_1(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1)[S_{wake}(\vec{k}_1)]
\]

\[
+ \iint d\vec{k}_1 d\vec{k}_2 F_2(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1, \vec{k}_2)[S_{wave}(\vec{k}_1)S_{wave}(\vec{k}_2)]
\]

\[
+ \iint d\vec{k}_1 d\vec{k}_2 F_2(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1, \vec{k}_2)[S_{wake}(\vec{k}_1)S_{wave}(\vec{k}_2)]
\]

\[
+ \iint d\vec{k}_1 d\vec{k}_2 F_2(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1, \vec{k}_2)[S_{wave}(\vec{k}_1)S_{wake}(\vec{k}_2)]
\]

\[
+ \iint d\vec{k}_1 d\vec{k}_2 F_2(\vec{k}_{scat}, \vec{k}_{inc}, \vec{k}_1, \vec{k}_2)[S_{wake}(\vec{k}_1)S_{wave}(\vec{k}_2)]
\]
Potential ship classification capability based on wake signature

**Ship A**
- Ship heading = 135 deg
- Ship speed = 12.5 m/s
- Wind speed = 7.65 m/s
- Wind direction = 10 deg

Power spectral density (dB)

**Ship B**
- Ship heading = 135 deg
- Ship speed = 12.5 m/s
- Wind speed = 7.65 m/s
- Wind direction = 10 deg

Power spectral density (dB)

**Ship A**
- Ship heading = 135 deg
- Ship speed = 12.5 m/s
- Wind speed = 7.65 m/s
- Wind direction = 170 deg

Power spectral density (dB)

**Ship B**
- Ship heading = 135 deg
- Ship speed = 12.5 m/s
- Wind speed = 7.65 m/s
- Wind direction = 170 deg

Power spectral density (dB)
Waves in ice fields: Boundary conditions at the ocean surface

**Hydrodynamic**
- free surface (X)
- two fluid, uniform shear surface
- stratified fluid rigid lid
- mass-loaded surface
- viscous layer
- viscoelastic layer
- visco-elastic-brittle layer ..... 

**Electromagnetic**
- perfect electrical conductor
- imperfect conductor
- PEC with infinitesimal dielectric coating
- PEC with finite thickness dielectric coating
- anisotropic impedance surface
- nonlinear impedance surface

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(a) Integration path closed by a semicircular arc at infinity in the lower halfplane.
(b) Integration path wrapped around fundamental branch cuts and poles assuming a lossless upper halfspace.
Dispersion relations for various sea ice models

Free surface \[ \omega^2 = g \kappa \tanh \kappa H \equiv gZ \]

Mass loading \[ \omega^2 = \frac{gZ}{1 + \frac{\rho_{\text{ice}}}{\rho}hZ} \]

Viscous layer \[ \omega^2 = \left( 1 - \frac{h \omega^2}{g \rho} \right) + \frac{\frac{\rho_{\text{ice}} - \rho}{\rho} (\omega^2 - 4i \kappa^2 \omega v)}{\omega^4 + 16 \kappa^4 \omega^2 v^2} \] \[ gZ \]

Thin elastic plate \[ \omega^2 = \frac{\left( g + \frac{L}{\rho} \kappa^4 \right) Z}{1 + \frac{\rho_{\text{ice}}}{\rho}hZ} \]

Viscoelastic layer \[ \omega^2 = Q \, gZ \]

where \[ Q = 1 + \frac{\rho_{\text{ice}}}{\rho} \cdot \frac{g^2 \kappa^2 S_{\kappa} S_{\alpha} - (N^4 + 16 \kappa^6 \alpha^2 v_e^4) S_{\kappa} S_{\alpha} - 8 \kappa^3 \alpha v_e^2 N^2 (C_{\kappa} C_{\alpha} - 1)}{g \kappa (4 \kappa^3 \alpha v_e^2 S_{\kappa} C_{\alpha} + N^2 S_{\alpha} C_{\kappa} - g \kappa S_{\kappa} S_{\alpha})} \]

\( H \) is the water depth
\( h \) is the ice thickness
\( L \) is the flexural rigidity
\( E \) is Young’s modulus
\( G \) is the effective shear modulus
\( P \) is the Poisson ratio
\( \nu_e \) is the effective kinematic viscosity
\( N = \omega + 2i \kappa^2 \nu_e \)
\( \alpha^2 = \kappa^2 - i \frac{\omega}{\nu_e} \)
\( \nu_e = \nu + \frac{iG_{\text{ice}}}{\rho_{\text{ice}} \omega} \)
Dispersion relations for free surface and pancake ice (viscous not shown)
Consequences of the changed dispersion in the presence of small ice floes

Computed for monostatic scattering geometry

NB: The lower figure is not a filtered version of the upper: it shows Doppler spectra from different wave species which happen to have identical spatial spectra.
When all else fails, ask a computer

Given a vast database of directional wave spectra over contiguous volumes of space and time, can a neural network:

- discover the patterns that reflect the operation of the source terms?

- reconstruct the associated mathematical operators in forms that can be applied to wave modelling and forecasting?

- provide a report in the language of physics that explains why the proposed functions were selected to embody the observed behavior?
Conclusion

HF radar has some unique capabilities, not all of which have been fully exploited.

The task of establishing the detailed functional forms of the source terms in the Hasselmann equation (or its more general relatives) is a huge challenge – as this audience knows well.

Perhaps the embryonic ideas outlined in this talk will lead to better ideas that bear fruit.