On Domination of Nonlinear Wave Interaction in the Energy Balance of Wind-Driven Sea

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This is the Hasselmann kinetic equation:

$$\frac{dN}{dt} = S_{nl} + S_{in} + S_{dis}$$

Three questions:

1. What is exactly N_k?

2. What is the structure of general solution for equation

$$S_{nl} = 0?$$

3. Which term is the most important in the balance equation

$$S_{nl} + S_{in} + S_{dis} = 0?$$

The answer to the last question is: S_{nl} is the most important term.

Shape of the surface is: $\eta(\vec{r}, t), \quad r = (x, y)$

Fluid is incompressible: $div\, \vec{V} = 0$

Flow is potential: $V = \nabla\, \Phi, \quad \Delta\Phi = 0, \quad \Phi = \Phi(\eta, z)$

Potential on the surface is: $\Psi = \Phi(\vec{r}, \eta(\vec{r}, t))$

Ψ_k and η_k are Fourier transforms of Ψ and η

We will use normal variable

$$a_k = \frac{1}{\sqrt{2}} \left\{ \left(\frac{g}{A_k} \right)^{1/4} \eta_k - i \left(\frac{A_k}{g} \right)^{1/4} \Psi_k \right\}$$

Here $A_k = k \tanh kH$, H is fluid depth, and

$$\omega_k = \sqrt{g A_k} = \sqrt{gk \tanh kH}$$

The total energy of the fluid, $H = T + U$, can be expanded in Taylor series

$$H = H_0 + H_1 + H_2 + \cdots$$

$H_0 = \int \omega_k |a_k|^2 dk$, H_1 is cubic in a_k, a_k^* and H_2 is quartic in a_k, a_k^*.

$$H_1/H_0 \sim \mu \qquad H_2/H_0 \sim \mu^2$$

The squared average steepness

$$\mu^2 = \frac{<\eta>^2 \ \omega_p^4}{g^2}$$

η and Ψ are canonical variables

$$\frac{\partial \eta_k}{\partial t} = \frac{\delta H}{\delta \Psi_k^*}, \qquad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta_k^*}$$

Statistical description:

$$n_{k\omega}\,\delta_{k-k'}\,\delta_{\omega-\omega'} =< a_{k\omega}\,a^*_{k',\omega'} >$$

Spectrum of elevation:

$$Q_{k\omega}\,\delta_{k-k'}\,\delta_{\omega-\omega'} =< \eta_{k\omega}\,\eta_{k',\omega'} >$$

$$Q_{k\omega} = \frac{\omega_k}{2}(n_{k\omega} + n_{-k,-\omega})$$

$$n_k\,\delta_{k-k'} =< a_k\,a^*_{k'} > \qquad Q_k\,\delta_{k-k'} =< \eta_k\,\eta_{k'} >$$

Spatial spectra:

$$n_k = \int_{-\infty}^{\infty} n_{k\omega}\,d\omega$$

$$Q_k = \frac{\omega_k}{2}(n_k + n_{-k}) = \int_{-\infty}^{\infty} Q_{k\omega}\,d\omega$$

Are the following statements correct?

1. $n_k = \frac{2}{\omega_k}\int_0^{\infty} Q_{k\omega} d\omega$
2. n_k satisfies the Hasselmann equation

Actually these statements are not correct.

In fact:

$$\frac{2}{\omega + k}\int_0^\infty Q_{k\omega} d\omega = N_0(k) + N_1(k) + \cdots$$

$$\frac{N_1(k)}{N_0(k)} \sim \mu^2 \left(\frac{k}{k_p}\right)^{3/2} \sim \mu^2 \left(\frac{\omega}{\omega_1}\right)^3$$

To find $N_1(k)$ one should perform a canonical transformation excluding cubic term H_1 in the Hamiltonian.

$$a_k = \frac{1}{\sqrt{2}}(q_k + ip_k)$$

Here $q_k, \quad p_k$ are canonical variables

$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*} \qquad \frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*}$$

Recall that

$$\frac{\partial a_k}{\partial t} + \imath \frac{\delta H}{\delta a_k^*} = 0$$

We need to perform canonical transformation excluding slave harmonics:

$$p_k, q_k \to R_k, \xi_k$$

The generating functional of transformation is

$$S = S[R_k, q_k]$$

and the implicit equations for R_k, ξ_k are

$$p_k = \frac{\delta S}{\delta q_k^*} \qquad \xi_k = \frac{\delta S}{\delta R_k^*}$$

The generating function S has the following form:

$$\begin{aligned} S \; = \; & \int R_k \, q_k^* \, dk + \frac{1}{2} \int A_{kk_1k_2} \, q_k \, q_{k_1} \, R_{k_2} \, \delta(k + k_1 + k_2) \, dk dk_1 dk_2 + \\ & + \frac{1}{3} \int B_{kk_1k_2} \, R_k \, R_{k_1} R_{k_2} \, \delta(k + k_1 + k_2) dk dk_1 dk_2 \end{aligned}$$

The coefficients of generation function:

$$A_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0+L_1-L_2}{\omega_0+\omega_1-\omega_2}\right) + \frac{1}{4}\left(\frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2} + \frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2}\right)$$

$$B_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2}\right) - \frac{1}{4}\left(\frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2} + \frac{L_2-L_0-L_1}{\omega_2-\omega_0-\omega_1}\right)$$

Here

$$L_0 = L_{kk_1k_2}, \quad L_1 = L_{k_1kk_2}, \quad L_2 = L_{k_2kk_1}$$
$$\omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2}$$

The new normal variable b_k

$$b_k = \frac{1}{\sqrt{2}}\left(\left(\frac{g}{A_k}\right)^{1/4} \xi_k - i\left(\frac{A_k}{g}\right)^{1/4} R_k\right)$$

satisfies equation

$$\frac{\partial b_k}{\partial t} + i\omega_k\, b_k + \frac{i}{2}\int T_{kk_1k_2k_3}\, b^*_{k_1}\, b_{k_2}\, b_{k_3}\, \delta_{k+k_1-k_2-k_3}\, dk_1 dk_2 dk_3 = 0$$

$$L_0 = L_{kk_1k_2} = -\frac{g^{1/4} A_k^{1/4}}{A_{k_1}^{1/4} A_{k_2}^{1/4}} \left[(k_1 k_2) + A_{k_1} A_{k_2}\right]$$

$$L_1 = L_{k_1,k,k_2} \qquad L_2 = L_{k_2,k,k_1}$$

$$N_0(k)\, \delta(k - k') =< b_k\, b^*_{k'} >$$

$N_0(k)$ satisfies the Hasselmann equation!

$N_1(k)$ is an important correction for $\omega \geq 4\, \omega_p$

$$N_1(k) = \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2} \left(N_{k_1} N_{k_2} - N_k N_{k_1} - N_k N_{k_2}\right) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2)\, dk_1 dk_2 +$$

$$+ \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_{k_1} - \omega_k - \omega_{k_2})^2} \left(N_{k_1} N_{k_2} + N_k N_{k_1} - N_k N_{k_2}\right) \delta(\vec{k}_1 - \vec{k} - \vec{k}_2)\, dk_1 dk_2 +$$

$$+ \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}_2, \vec{k}, \vec{k}_1)|^2}{(\omega_{k_2} - \omega_k - \omega_{k_1})^2} \left(N_{k_1} N_{k_2} + N_k N_{k_2} - N_k N_{k_1}\right) \delta(\vec{k}_2 - \vec{k} - \vec{k}_1)\, dk_1 dk_2 +$$

$$+ \frac{1}{2} \int \frac{|V^{(0,3)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_k + \omega_{k_1} + \omega_{k_2})^2} \left(N_{k_1} N_{k_2} + N_k N_{k_1} + N_k N_{k_2}\right) \delta(\vec{k} + \vec{k}_1 + \vec{k}_2)\, dk_1 dk_2$$

II. Solutions of equation $S_{nl} = 0$

$$S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \, \delta(k + k_1 - k_2 - k_3) \, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times$$
$$\times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) dk_1 dk_2 dk_3.$$

Thereafter $H = \infty$. Let $|k_3| \ll |k|$, $|k_1 \ll |k|$

$$T_{kk_1k_2k_3} \simeq \frac{1}{2} k \, k_1^2 \, T_{\theta_1,\theta_3}$$
$$T_{\theta_1,\theta_2} = 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3)$$

Let $N = N_0(k) + N_1(k)$, $N_1(k) \ll N_0(k)$

N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij} \, k^2 \, \frac{\partial}{\partial k_j} N_1,$$

where D_{ij} is the tensor of diffusion coefficients,

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq \, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2 \, p_i p_j N(\theta, q) N(\theta_3, q)$$

If $\quad N_k = k^{-x}, \quad S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x)$

$F(x) < \infty, \quad$ if $\quad 5/2 < x < 19/2.$

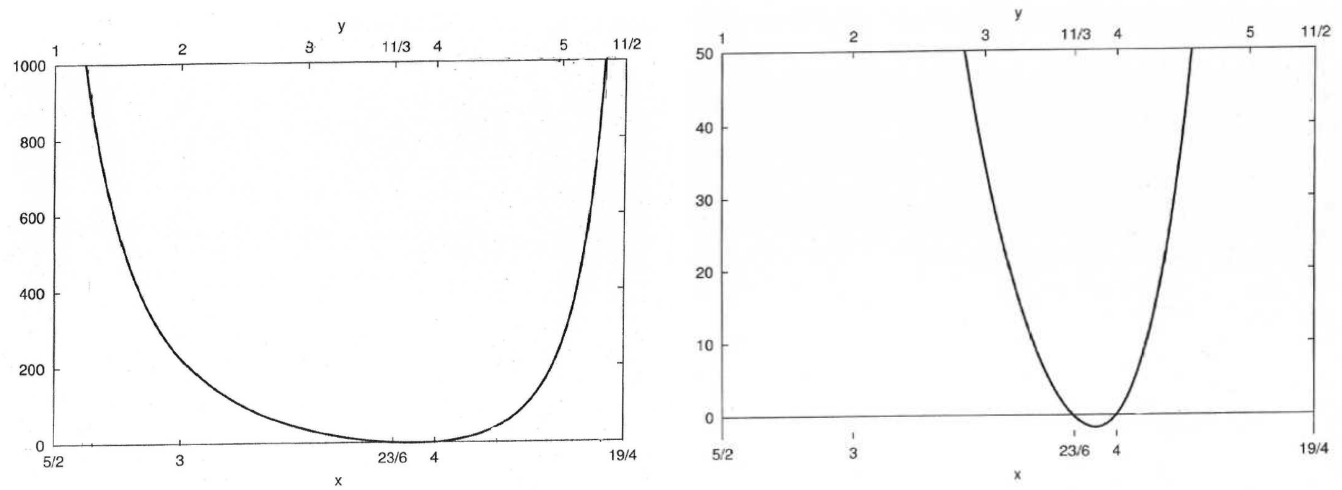

Figure 1: (a) Plot of function $F(x)$. (b) Plot of function $F(x)$: zoom in the vertical direction

On Figure 1 is presented plot of function $F(x)$ for isotropic case calculated numerically. $F(x)$ has two zeros at $x = y_1 = 4, \quad x = y_2 = 23/6$.

Corresponding solutions are KZ spectra:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4}, \qquad N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}$$

Here $\quad c_p = 0.219, \quad c_q = 0.227 \quad$ are Kolmogorov constants, $\quad P$ is flux of energy to $k \to \infty$, and Q is a flux of wave action to $k \to 0$.

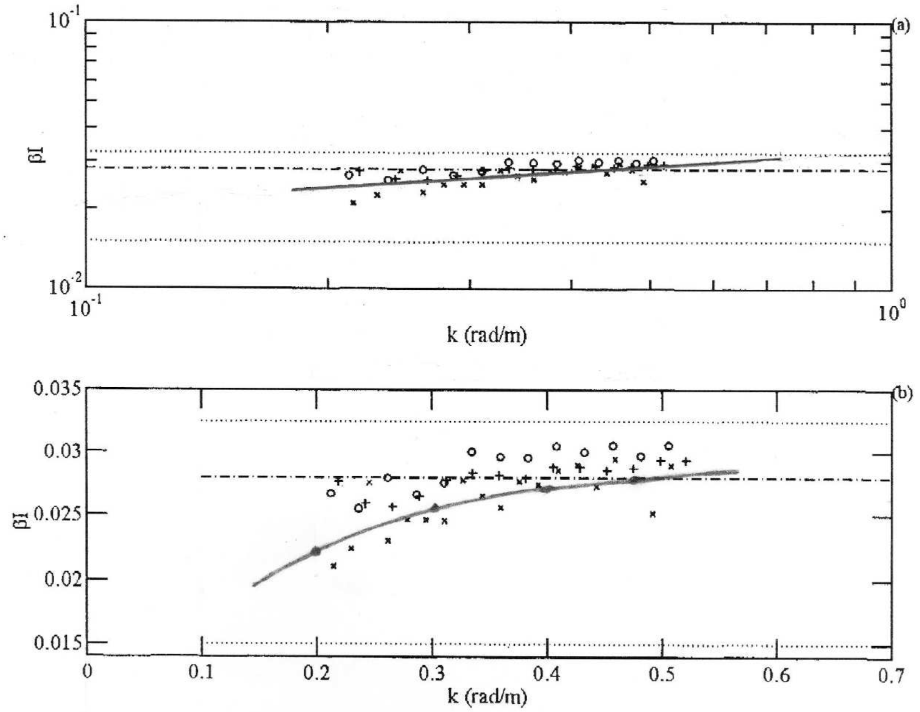

Figure 2: Dimensionless wavenumber spectral coefficient β_i plotted in logarithmic scales (a) and linear scales (b), taken from [1]. Here crosses represent omnidirectional (averaged by angles) spectrum and dots correspond to $\xi(k) = 2\beta_I \, u_* \, g^{-0.5} \, k^{-2.5}$. The solid line on (a) and solid curve on (b) correspond to $\xi(k) \simeq k^{-7/3}$.

A careful study of experimental results shows that in the majority of cases the spectral area right behind the spectral peak can be better approximated by tail $\omega^{-11/3}$ in the frequency spectrum and by tail $k^{-7/3}$ in the spatial spectrum.

[1] P.A. Hwang, W.C. Wang, E.J. Walsh, W.B. Krabill and R.N. Swift, Airborne measurements of the directional wavenumber spectra of ocean surface waves. Part 1. Spectral slope and dimensionless spectral coefficient. J. Phys. Oceanogr. 30 (2000b), 2753-2767.

A general anisotropic solution

$$N(\omega,\phi)\,d\omega d\phi = N(\vec{k})\,d\vec{k}$$

satisfies equation

$$\frac{\partial N(\omega,\phi)}{\partial t} = S_{nl}(\omega,\phi) = \left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) A$$

$$A(\omega,\phi) = \int_0^\infty d\omega' \int_0^{2\pi} d\phi'\, G(\omega,\omega',\phi-\phi')\, S_{nl}(\omega',\phi')$$

$G(\omega,\omega',\phi-\phi')$ is a Green function, then

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) G = \delta(\omega-\omega')\,\delta(\phi-\phi')$$

$A \to N$ one to one correspondent. The "diffusion approximation".

$$A = \frac{H_0}{g^4}\,\omega^{15}\,N^3, \quad H_0 \simeq 1.83$$

$$S_{nl} = 0 \quad \rightarrow \quad \left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A = 0$$

$$A|_{\omega=\omega_0} = A_0(\phi) = \sum_{n=2}^{\infty} A_n \, \cos n\phi$$

$$A(\omega, \phi) = A_0 + \frac{A_1}{\omega} \, \cos\phi + \sum_{n=2}^{\infty} A_n \left(\frac{\omega_0}{\omega} \right)^{-1/2+\sqrt{1/4+4n^2}} \, \cos n\phi$$

$A_0 = P$ is a flux of energy to $k \rightarrow \infty$

$A_1 = M$ is a flux of momentum to $k \rightarrow \infty$

Other terms describe angular spreading. If there is a flux of action Q from infinity, then

$$A \rightarrow A + \omega \, Q$$

Dominance of term S_{nl}

The nonlinear interaction term can be presented in the form

$$S_{nl} = F_k - \Gamma_k \, N_k,$$

where

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2 \, \delta(k+k_1-k_2-k_3) \, \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}) \, N_{k_1}N_{k_2}N_{k_3} \, dk_1dk_2dk_3$$

and Γ_k, the dissipation rate due to the presence of four-wave processes, is

$$G_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \, \delta(k+k_1-k_2-k_3) \, \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}) \times \\ \times (N_{k_1}N_{k_2} + N_{k_1}N_{k_3} - N_{k_2}N_{k_3}) \, dk_1dk_2dk_3$$

$$S_{in} = \gamma_{in}(k) \, N_k \qquad S_{dis} = -\gamma_{dis}(k) \, N_k$$

$$\gamma_k = \gamma_{in}(k) - \gamma_{dis}(k)$$

$$S_{nl} + \gamma(k) \, N_k = 0 \qquad N_k = \frac{F_k}{\Gamma_k - \gamma_k}$$

A rough estimate

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k} k^{10} N_k^2$$

$\Gamma_k, \quad F$ diverge on KZ spectra! As a result for the "mature sea", for the narrow in angle spectrum

$$N_k \simeq \frac{3}{2} \frac{E}{\sqrt{g}} \frac{k_p^{3/2}}{k^4} \theta(k - k_p)$$

we get equation

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta$$

that includes a huge enhancing factor $36\pi \simeq 113.04$. For the very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4}\omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta$$

For isotropic spectra

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2}\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4$$

the enhancing factor $45\pi/2 \simeq 70.65$.

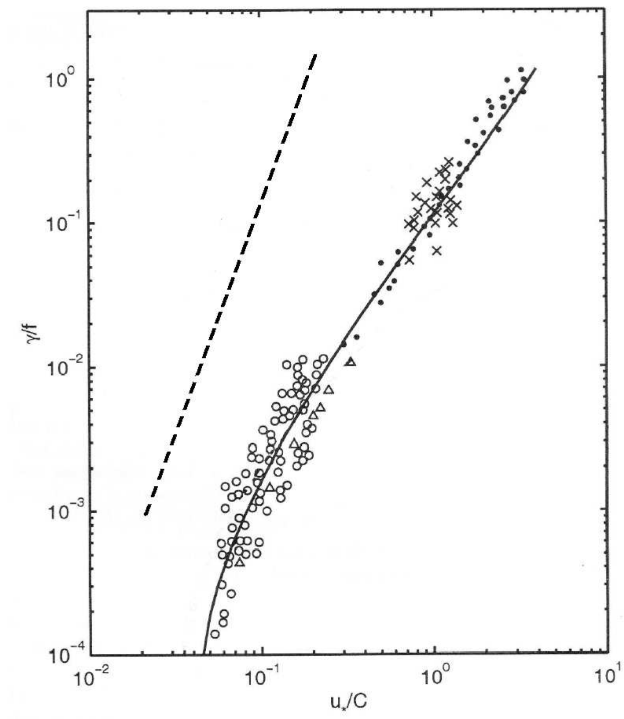

Figure 3: Comparison of experimental data for the wind-induced growth rate $2\pi\,\gamma_{in}(\omega)/\omega$ taken from [2] and the damping due to four-wave interactions $2\pi\,\Gamma(\omega)/\omega$, calculated for narrow in angle spectrum at $\mu \simeq 0.05$ using Eq. (1) (dashed line)

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta \qquad (1)$$

[2] S.J. Komen, L. Cavaleri, M. Donelan, K. Hasselmann, S. Hasselmann. P.A.E.M. Janssen. Dynamics and Modelling of Ocean Waves. Cambridge University Press, 1994.

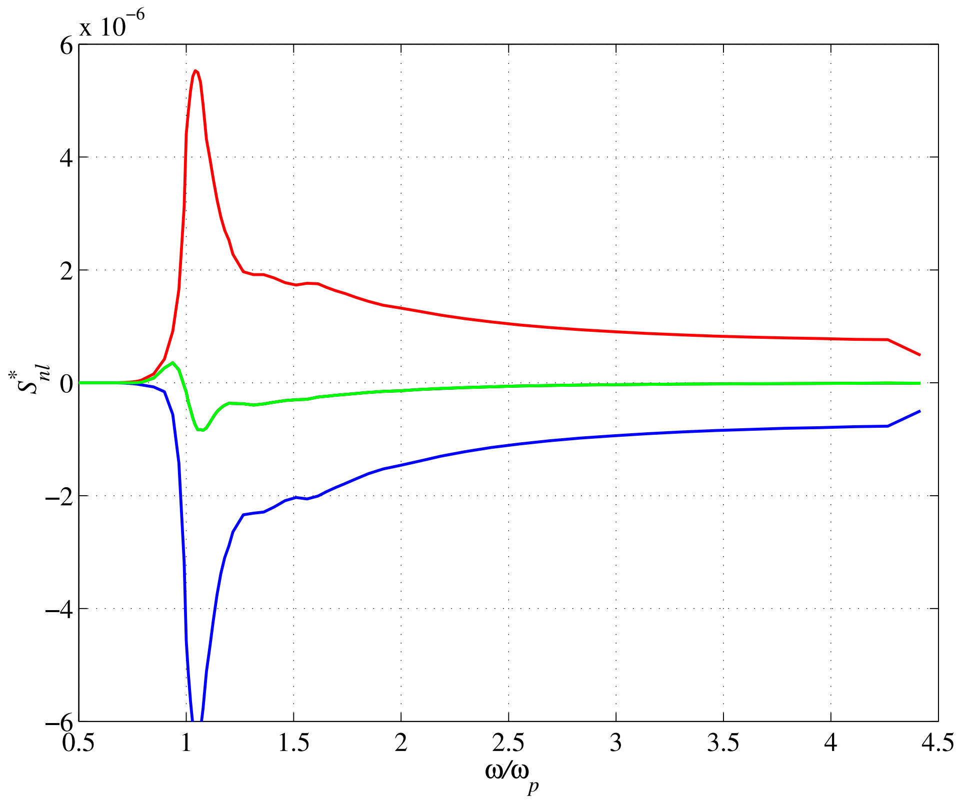

Figure 4: Split of nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k\, N_k$ (lower curve)

For the nonlinear interaction term $S_{nl} = F_k - \Gamma_k\, N_k$ the magnitudes of constituents F_k and $\Gamma_k\, N_k$ essentially exceed their difference. They are one order higher than the magnitude of S_{nl}!

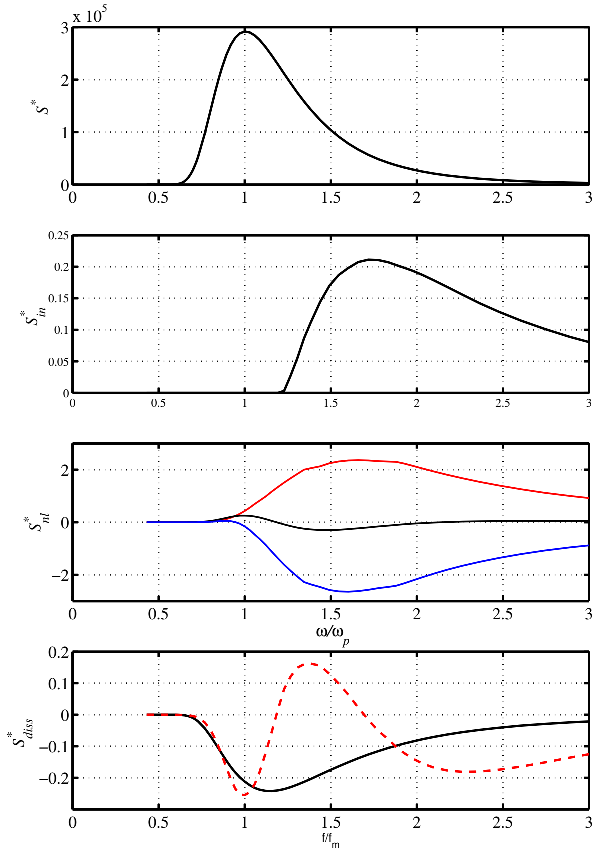

Figure 5: Plots from the article [3] and the split of S_{nl} (panel 3)

[3]Komen, G.J., S. Hasselmann S. and K. Hasselmann, On the existence of a fully developed wind-sea spectrum. J. Phys. Oceanogr. 14 (1984), 1271-1285.

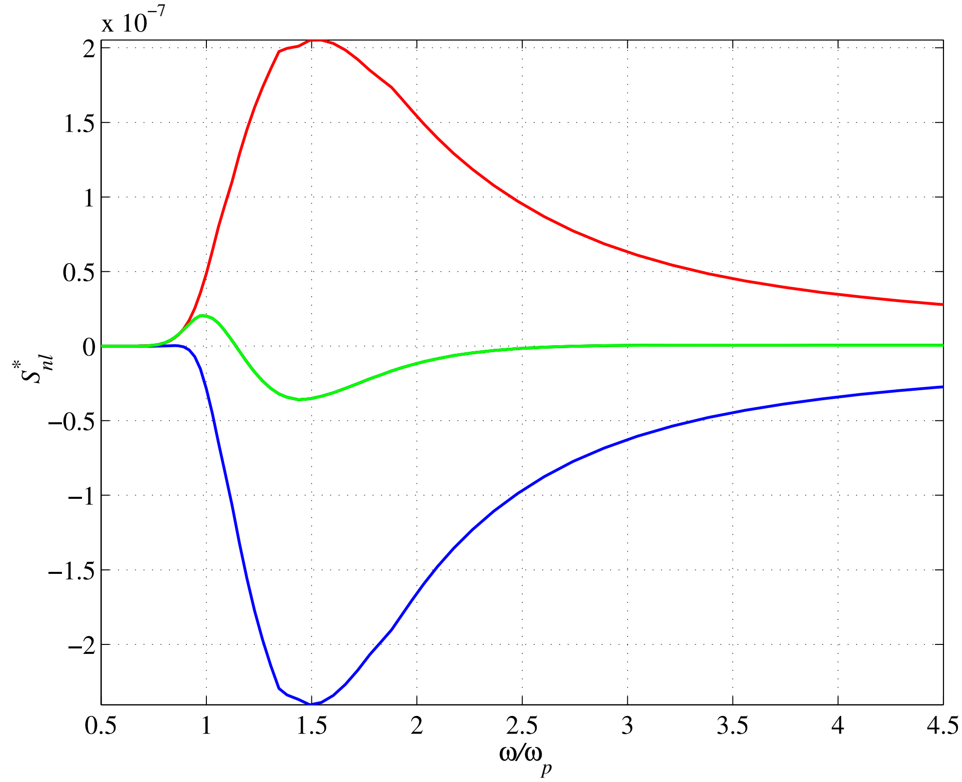

Figure 6: Zoom of panel 3

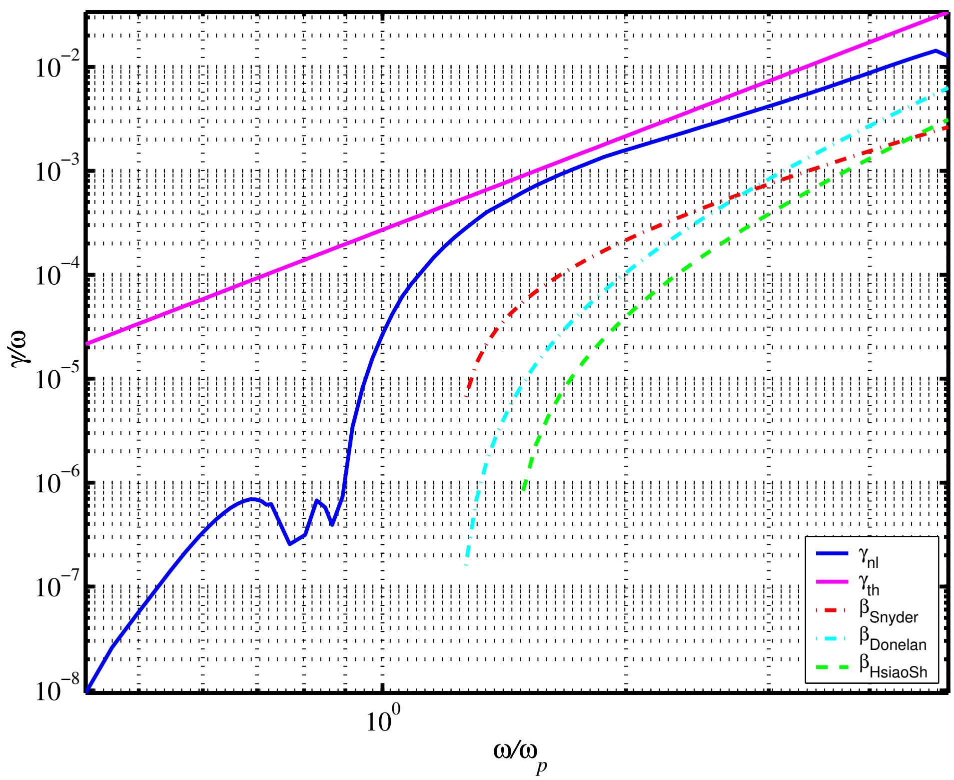

Figure 7: Γ_k compared with different models for S_{in}