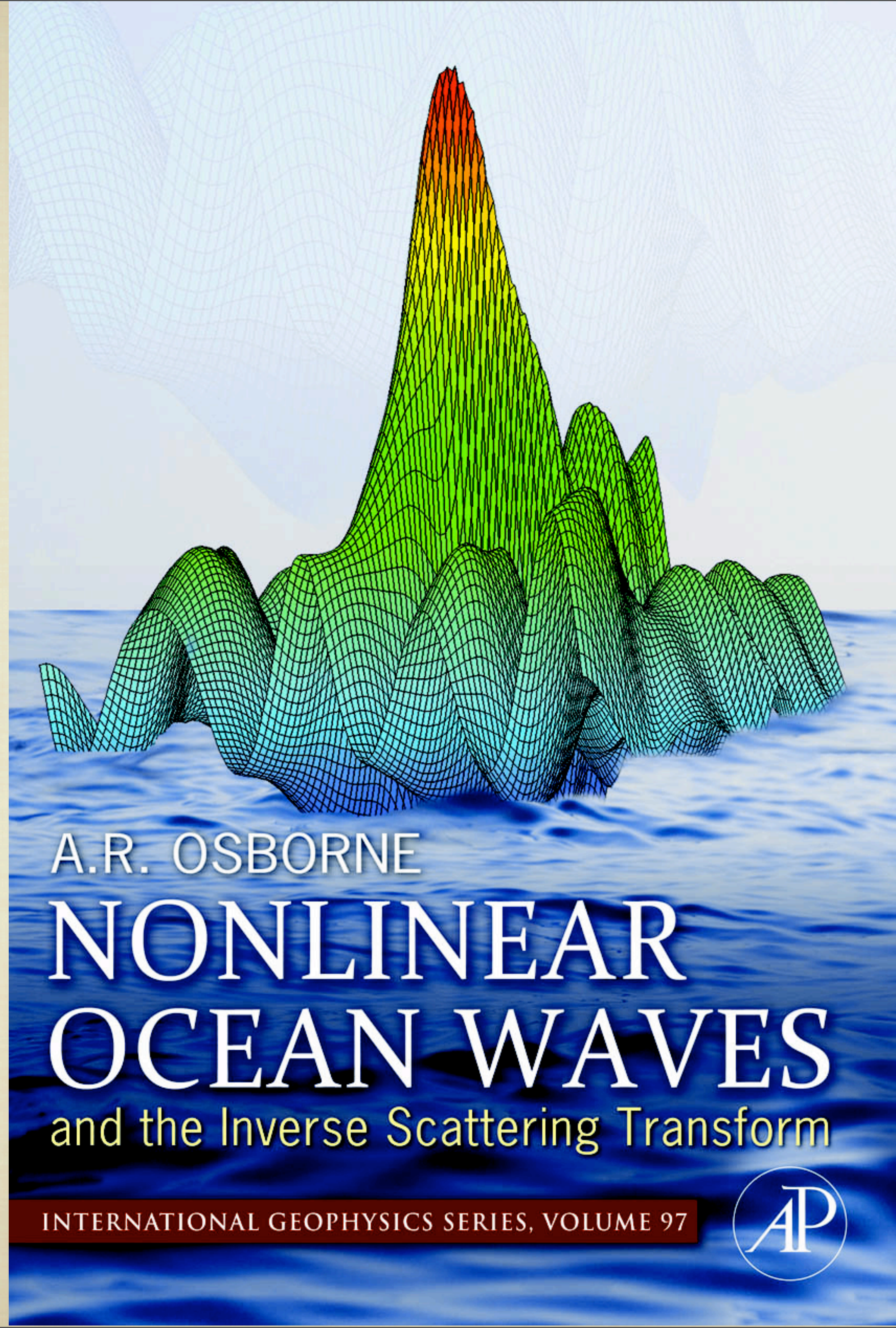
Modeling Waves in Shallow (and Deep) Water

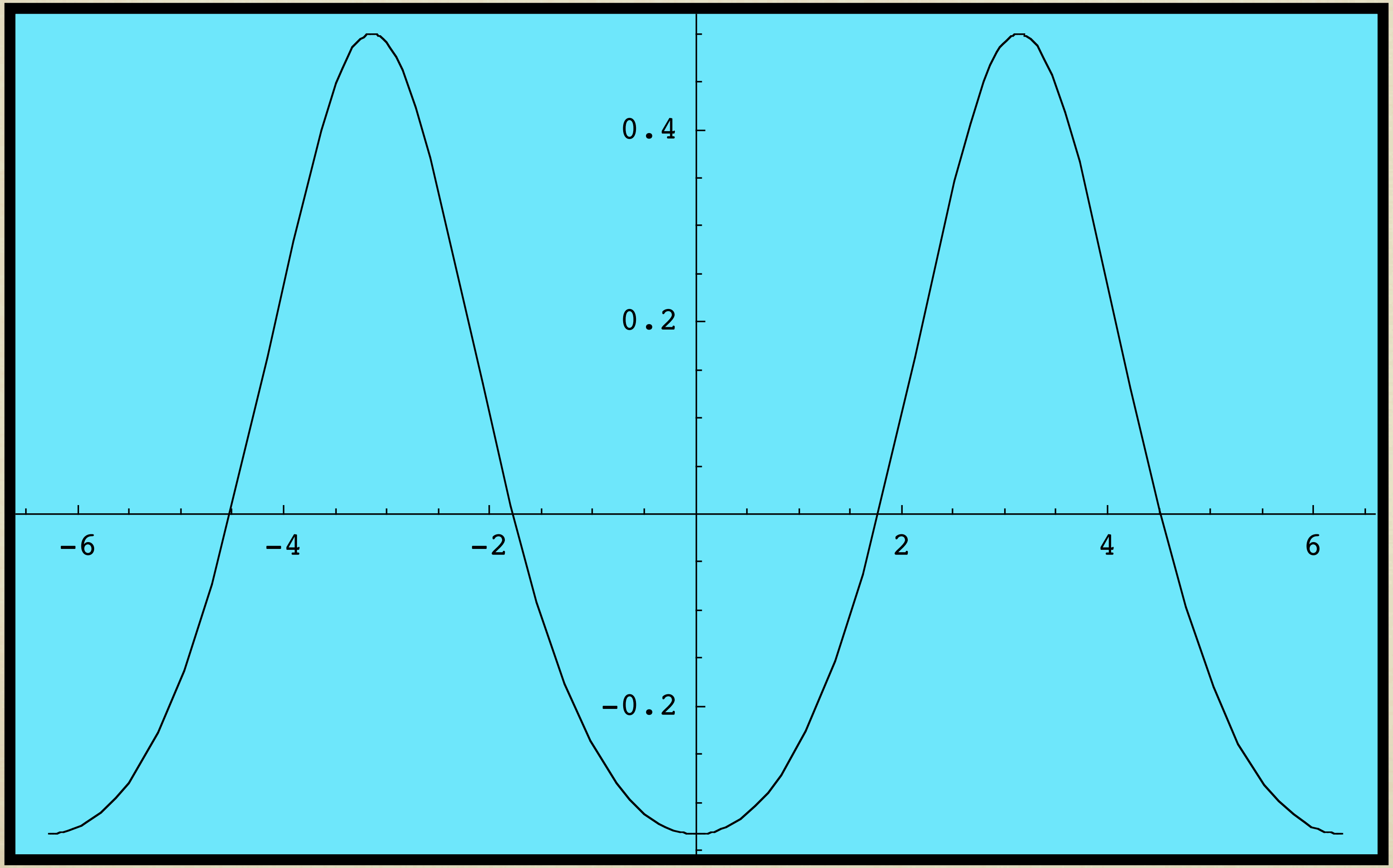
Al Osborne
Don Resio

FUNDERS: ONR AND ARMY CORP OF ENGINEERS

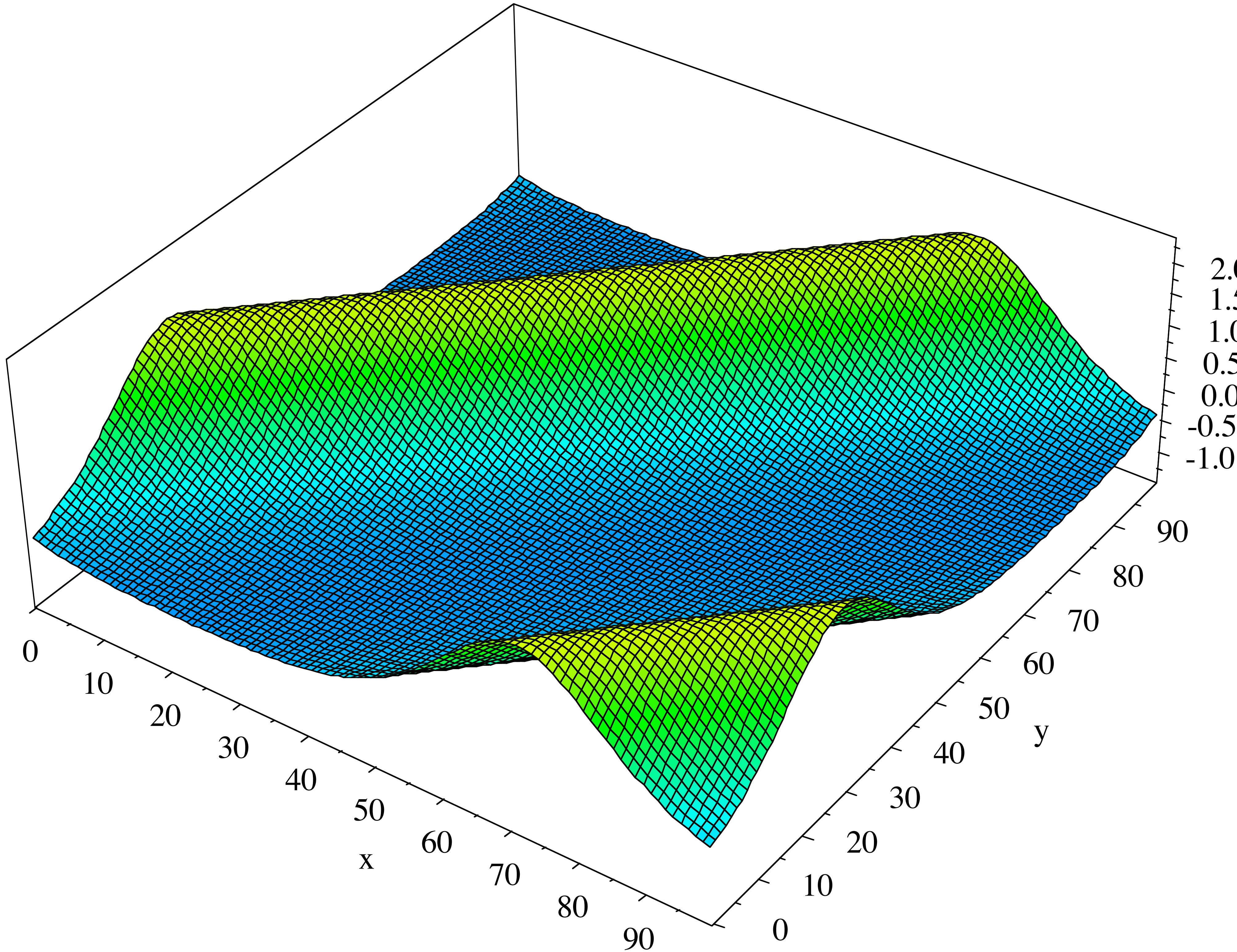

A.R. OSBORNE
NONLINEAR
OCEAN WAVES
and the Inverse Scattering Transform
INTERNATIONAL GEOPHYSICS SERIES, VOLUME 97
AP

900 pages

Stokes Wave Nonlinearity



Nonlinear Interactions for Directional Stokes Waves



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-1.0
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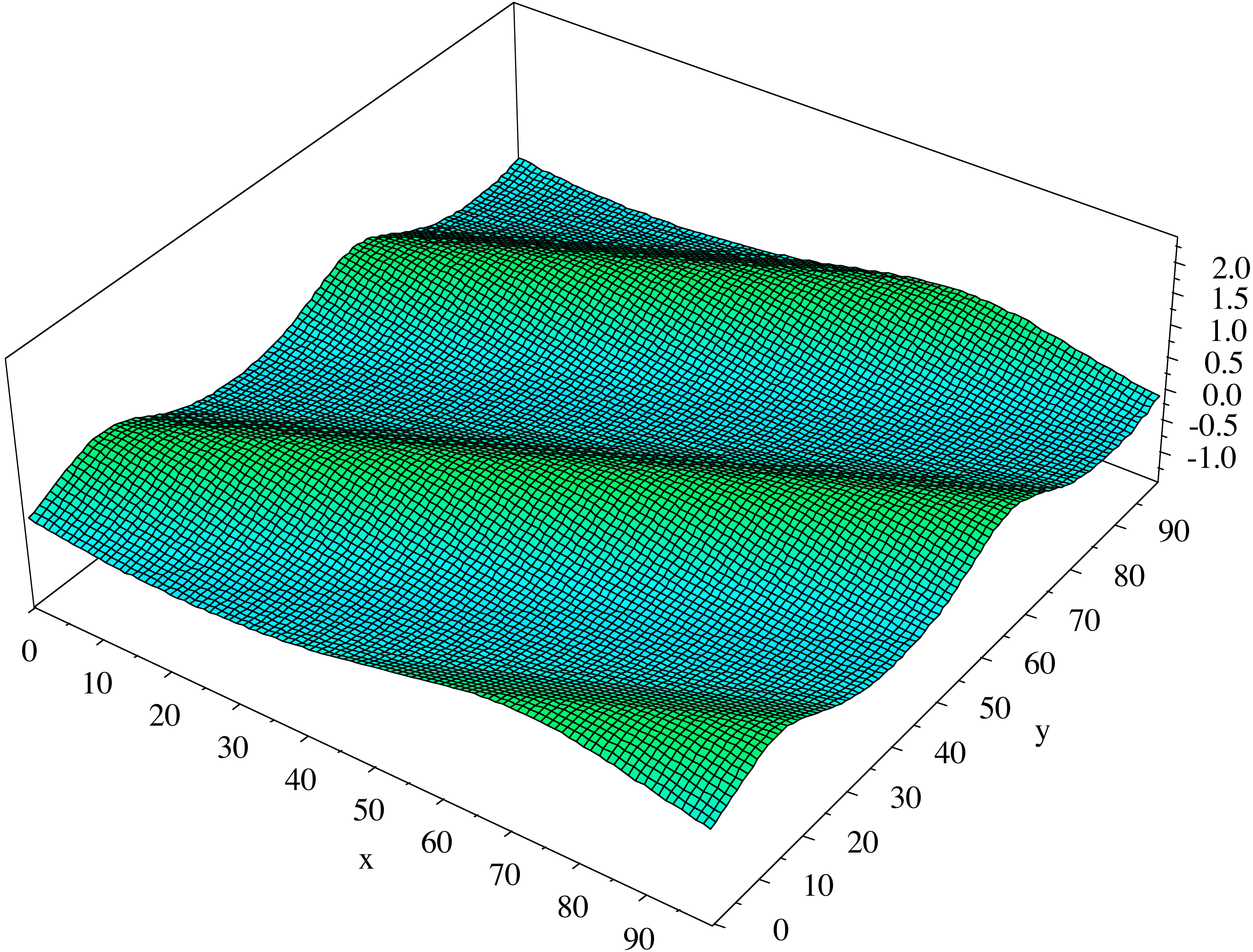
y



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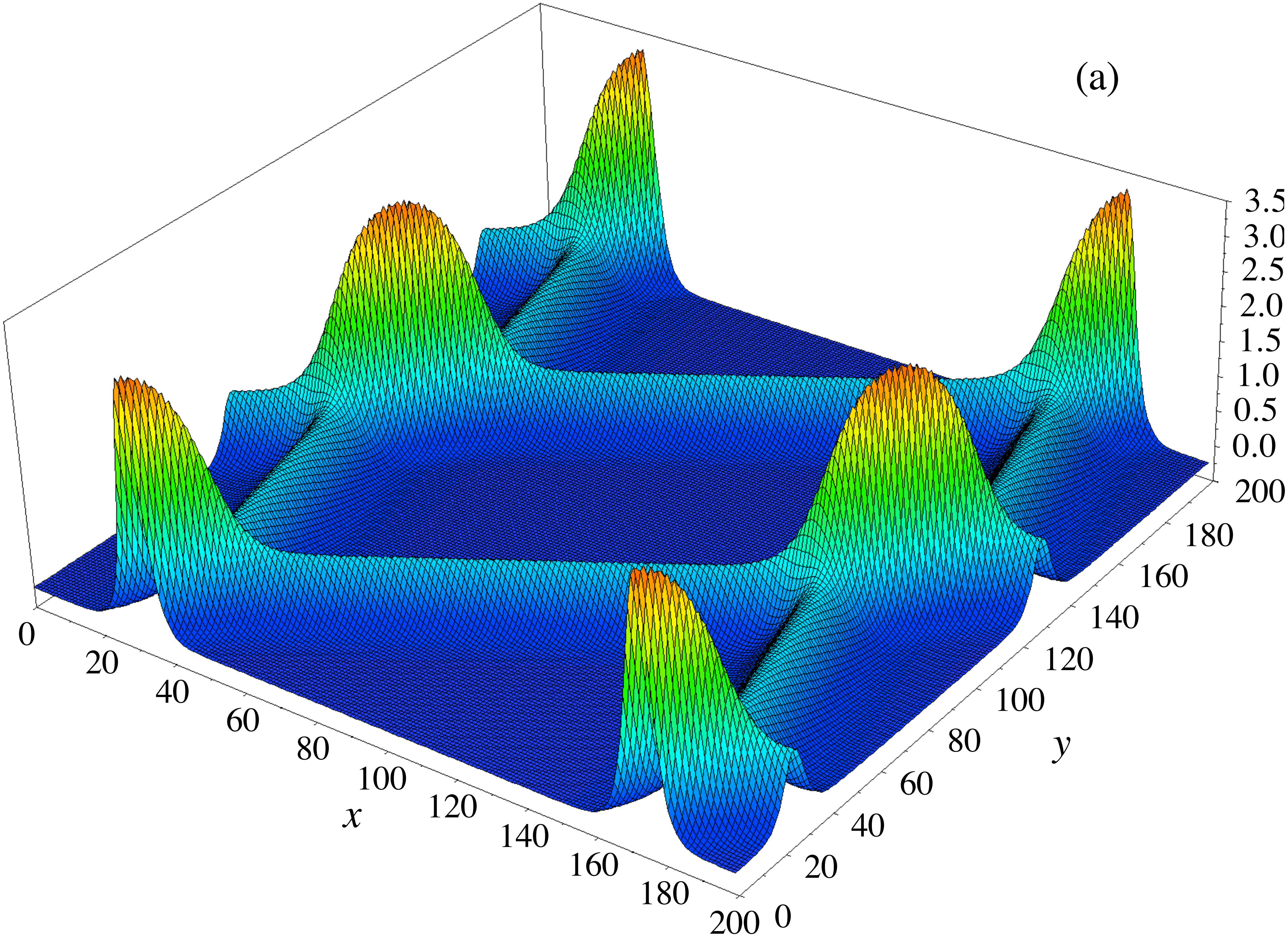
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1.5
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20
40
60
80
100
120
140
160
180
200
x
0
20
40
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80
100
120
140
160
180
200
y

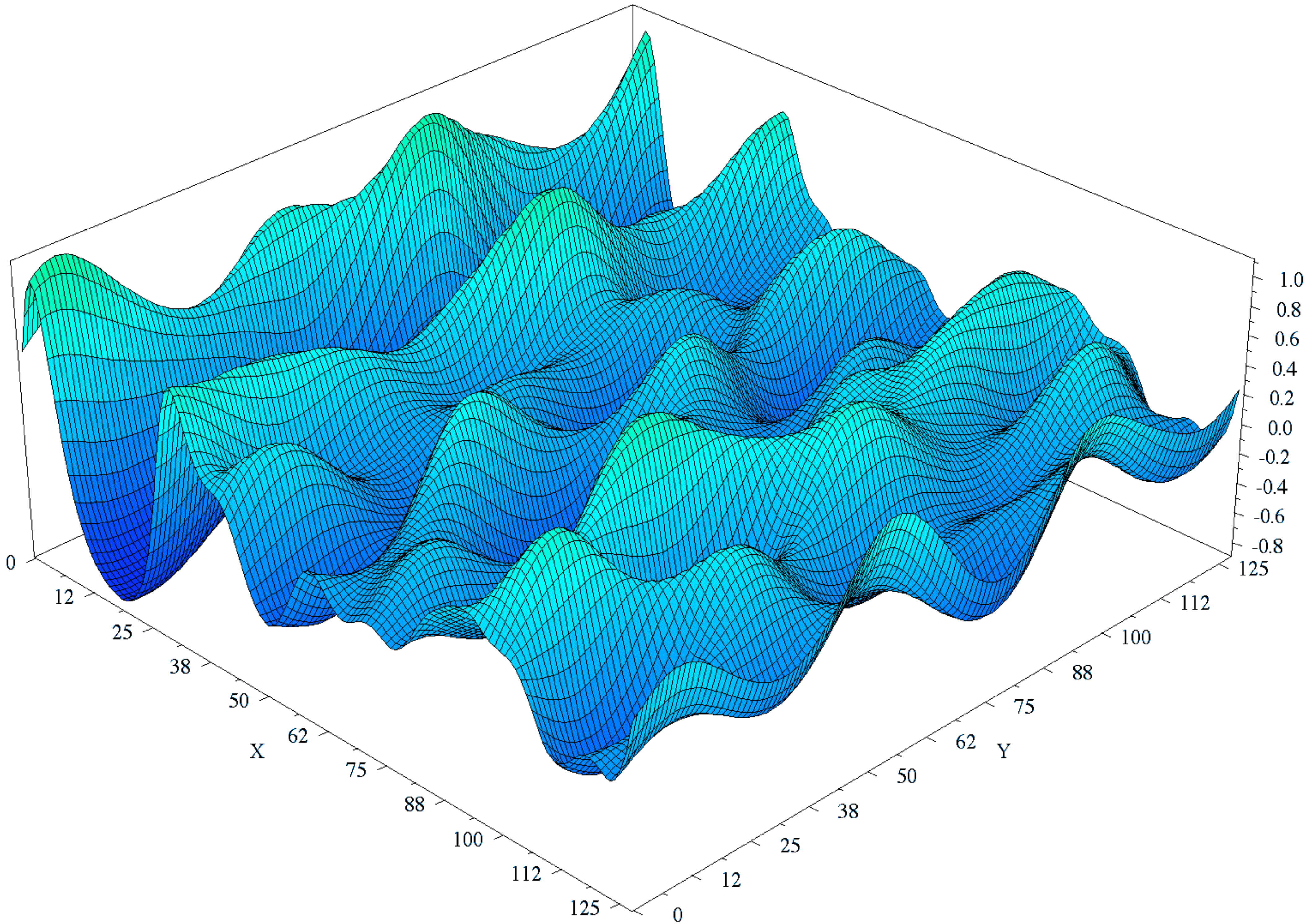

11. 9. 2001 08:42

Kadomtsev and Petviashvili (KP) equation

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \gamma\,\partial_x^{-1}\eta_{yy} = 0$$

$$\eta(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t)$$

$$\theta(x,y,t) = \sum_{\mathbf{m}\in\mathbf{Z}} e^{-\frac{1}{2}\mathbf{m}\bullet\mathbf{B}\mathbf{m}+i\mathbf{m}\bullet\mathbf{k}x+i\mathbf{m}\bullet\mathbf{l}y-i\mathbf{m}\bullet\boldsymbol{\omega}t+\mathbf{m}\bullet\boldsymbol{\phi}}$$



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0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
0
12
25
38
50
62
75
88
100
112
125
X
0
12
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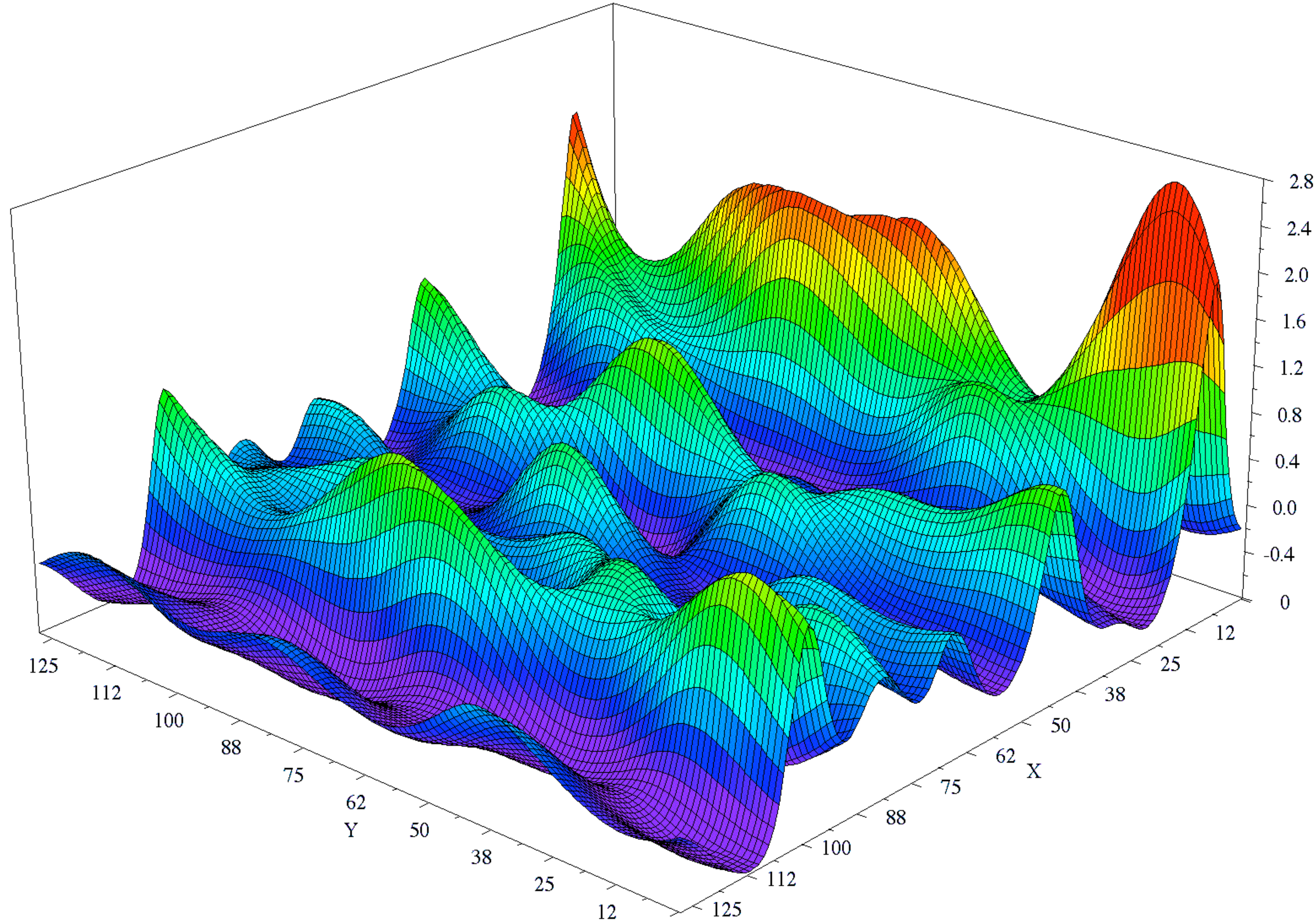
50
62
75
88
100
112
125
Y



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2.0
1.6
1.2
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0.0
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125
112
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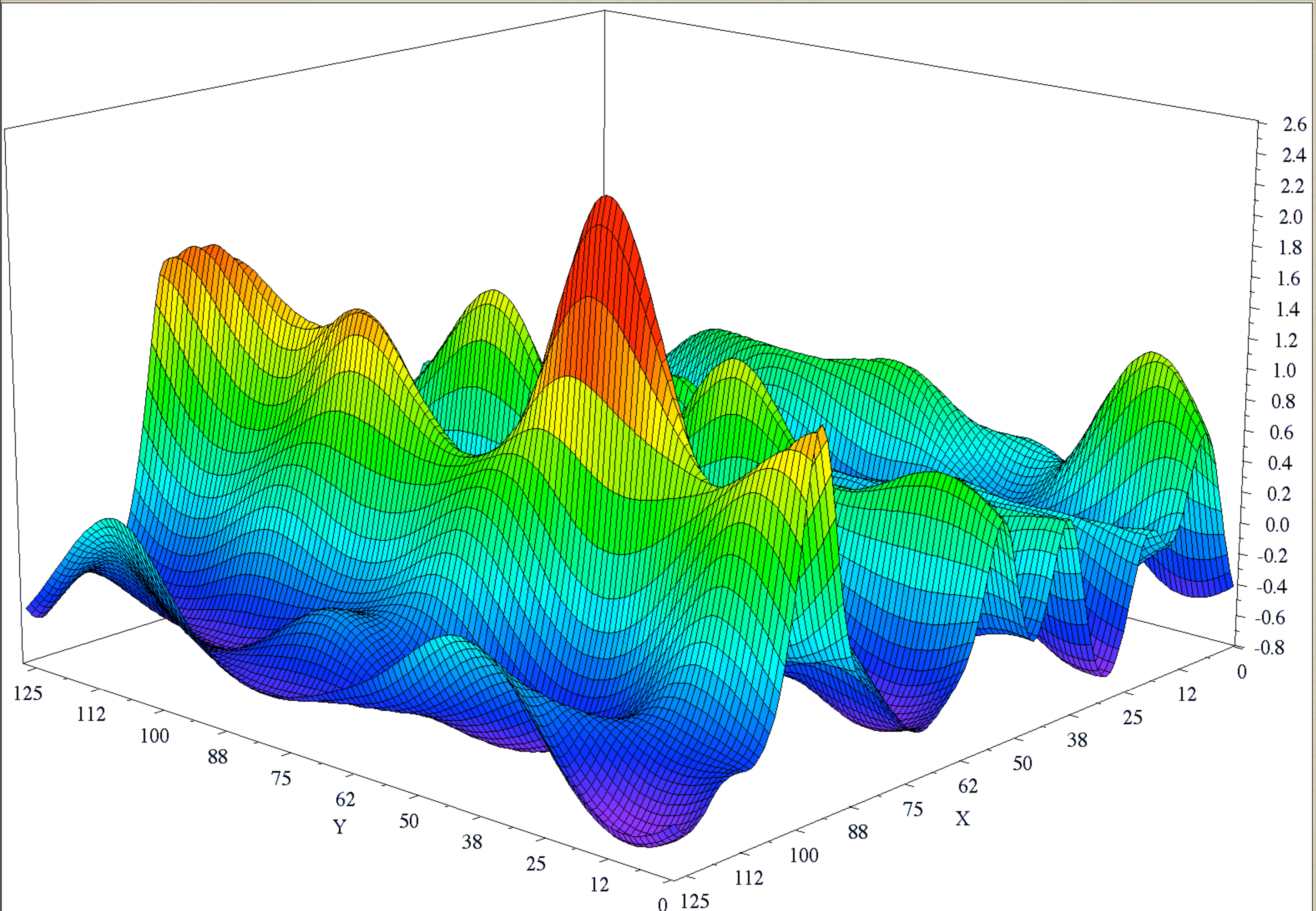
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125
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12
0

Modulational Instability

Davey-Stewartson Equations

$$i\Psi_\tau + \lambda\Psi_{XX} + \mu\Psi_{YY} + \chi|\Psi|^2\Psi = \chi_o\Psi\Phi_X$$

$$\Phi_{XX} + \Phi_{YY} = -\beta\left(|\Psi|^2\right)_X$$

$$\eta(x,y,t) \approx \frac{i\omega\sqrt{gk_o}}{gk_o^2}\Psi(x,y,t)e^{ik_ox-i\omega_ot} + c.c. + \ldots$$

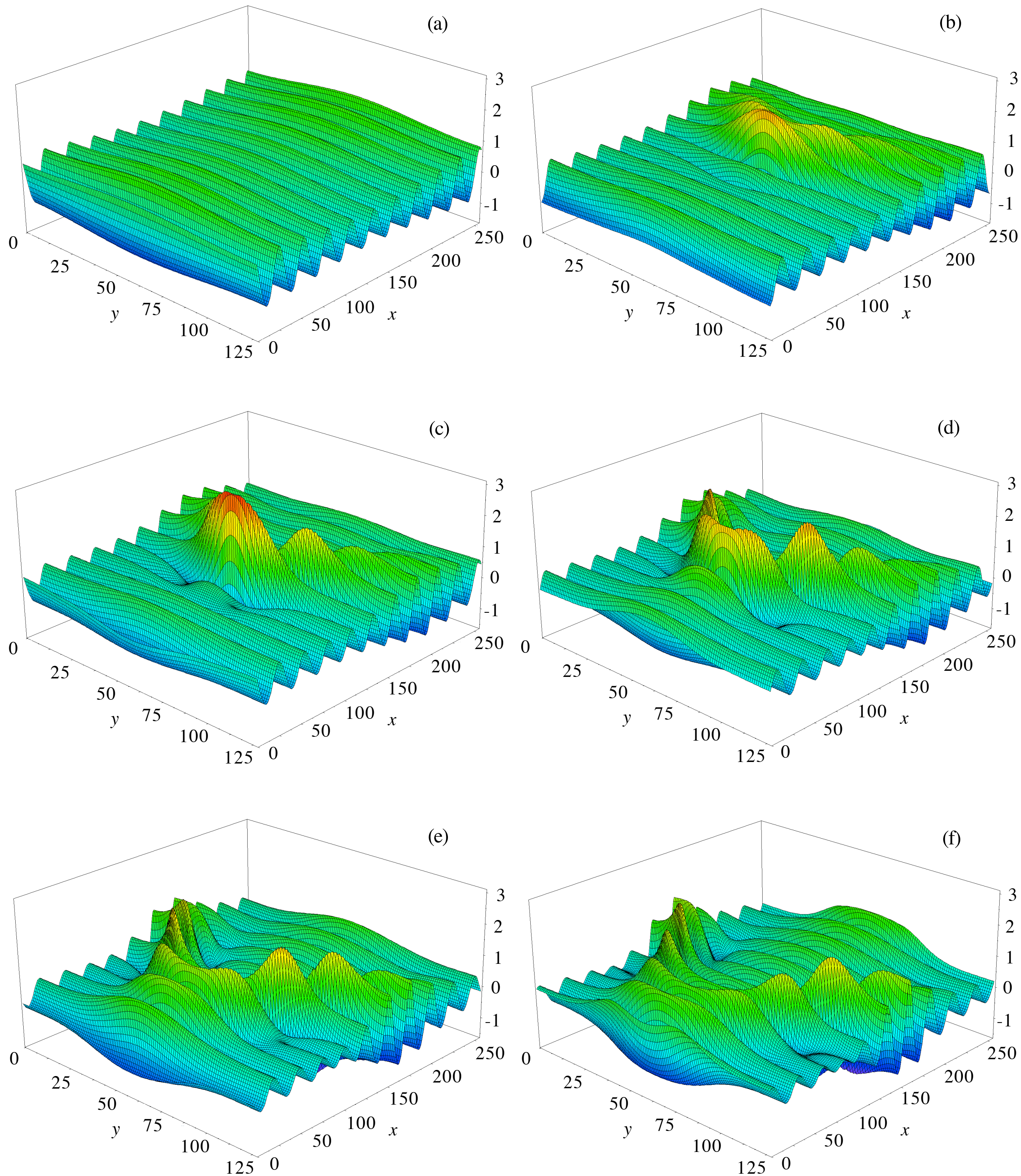
$$\phi(x,y,t) \approx \sqrt{\frac{g}{k^3}}\left(\Phi(x,y,t) + \frac{\cosh k(z+h)}{\cosh kh}\Psi(x,y,t)e^{ikx-i\omega t} + c.c.\right)$$

Instabilities in Deep Water Wave Trains

Deep water wave trains have ***two kinds of waves*** (at leading order):

(1) Stokes waves - quasi linear wave trains described by a spectrum.
(2) Unstable wave packets governed by Benjamin-Fier instability.

Threshold - Only ***Stokes waves*** occur below the BF instability threshold. When the waves get big enough, rapidly enough ***unstable packets*** begin to form. The extreme case is called a ***rogue sea***.



(a)
(b)
(c)
(d)
(e)
(f)
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0
-1
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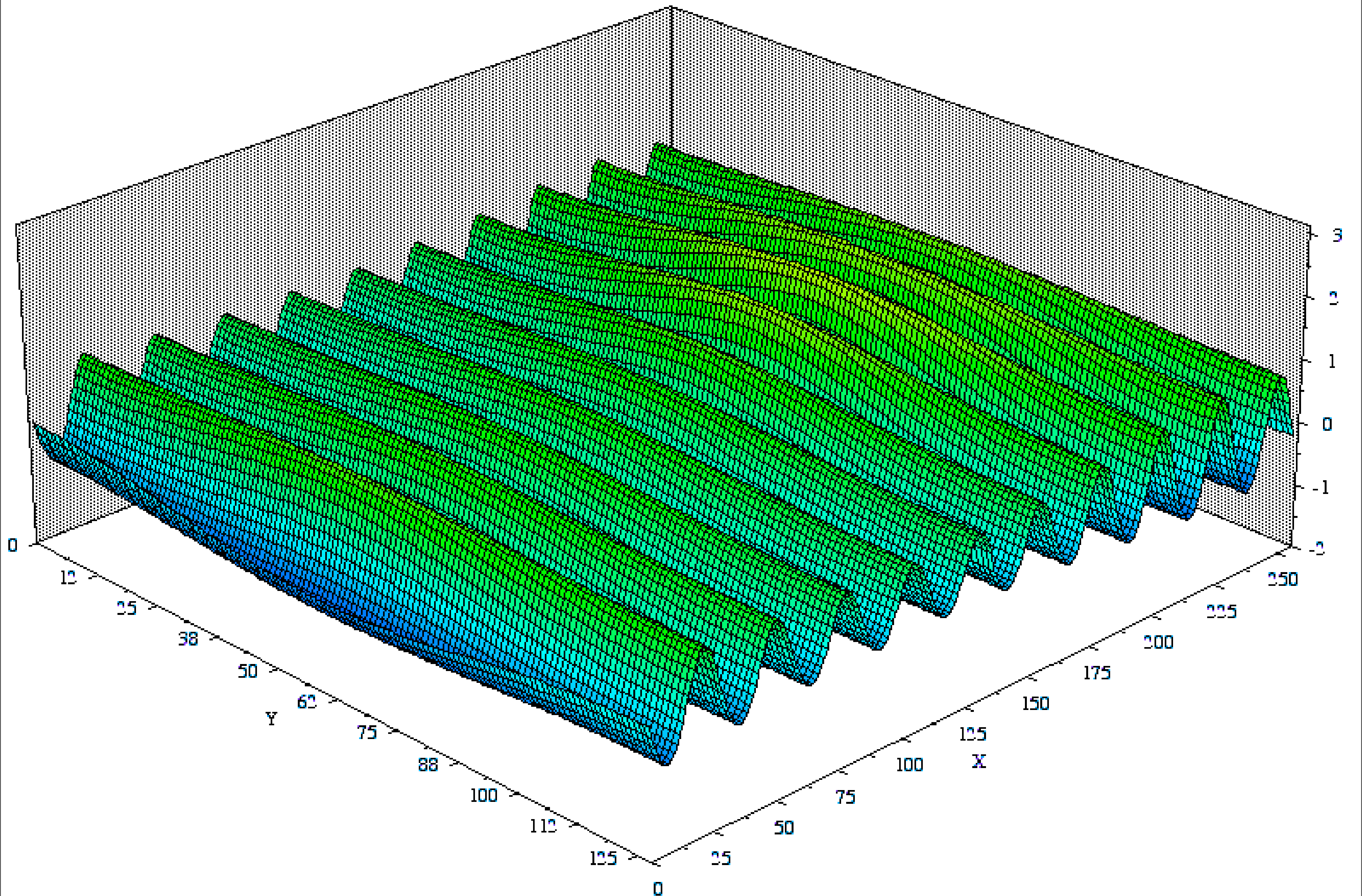
25
50
75
100
125
y
0
50
100
150
200
250
x


3
2
1
0
-1
-2
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13
25
38
50
63
75
88
100
113
125
Y
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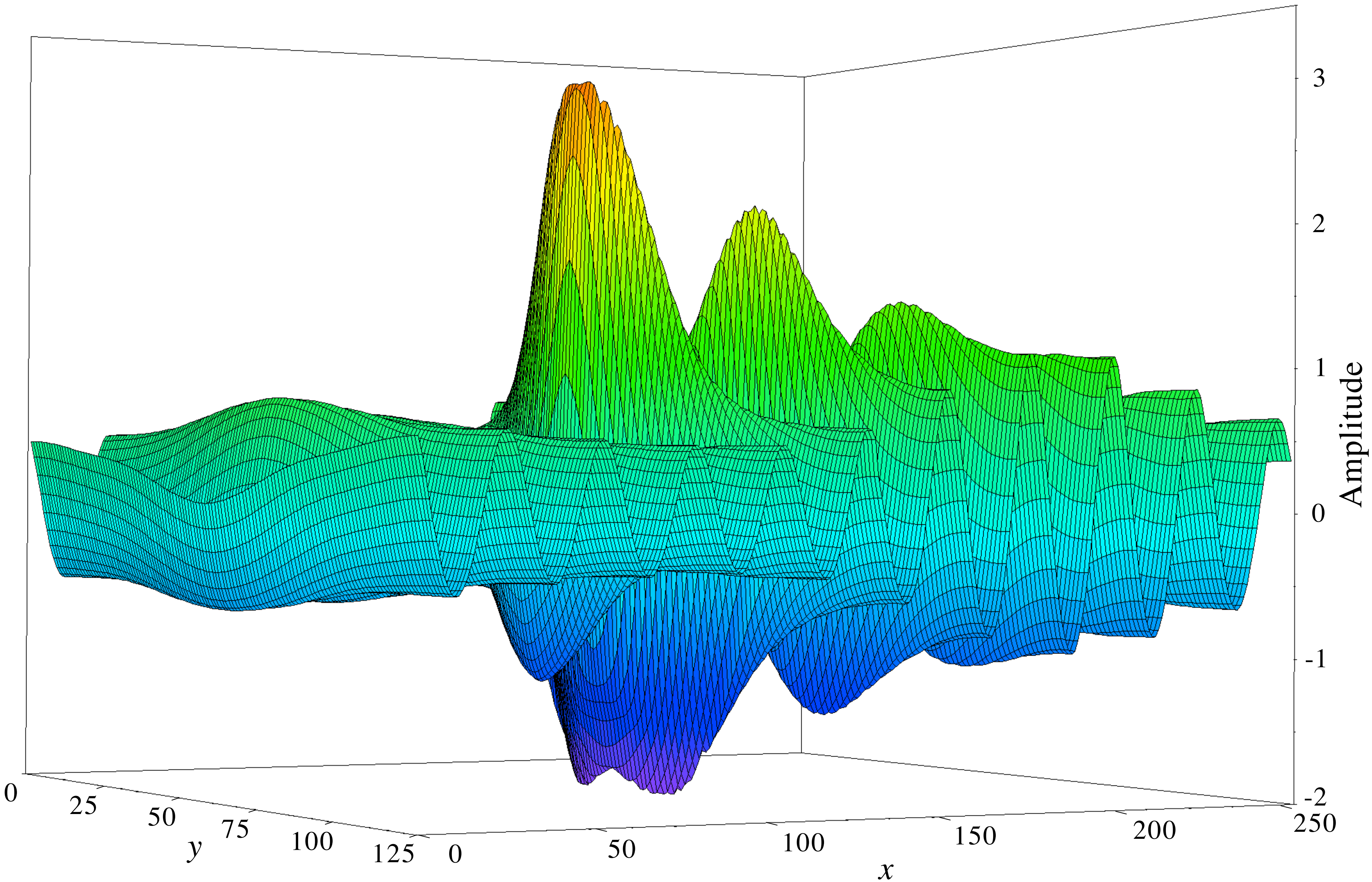
25
50
75
100
125
150
175
200
225
250
X



3
2
1
0
-1
-2
Amplitude
0
25
50
75
100
125
y
0
50
100
150
200
250
x

Conclusions:

Nonlinear Fourier Analysis:
The Inverse Scattering Transform for Periodic Boundary Conditions

(1) Physics of nonlinear waves as a nonlinear Fourier decomposition of nonlinear wave trains in terms of nonlinear Fourier components, Including **sine waves, Stokes waves, solitons, holes, unstable (rogue) Modes, shock waves, tabletop solitons, vortices, etc**.

(2) IST gives us approaches for **analyzing directional data** to obtain the nonlinear Fourier spectrum

(3) Hyperfast numerical methods based upon IST are **three orders of magnitude faster** than FFT approaches.

(4) Perfectly parallel algorithms give us many more orders of magnitude speed up un computers with thousands of processors.

Deterministic Waves: How Linear Fourier Analysis Works

Given a linear wave equation (shallow water):

$$u_t + c_o u_x + \beta u_{xxx} = 0$$

$c_o = \sqrt{gh}$

$\beta = c_o h^2 / 6$

Fourier Transform:

$$u_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx$$

Time dependence of Fourier components

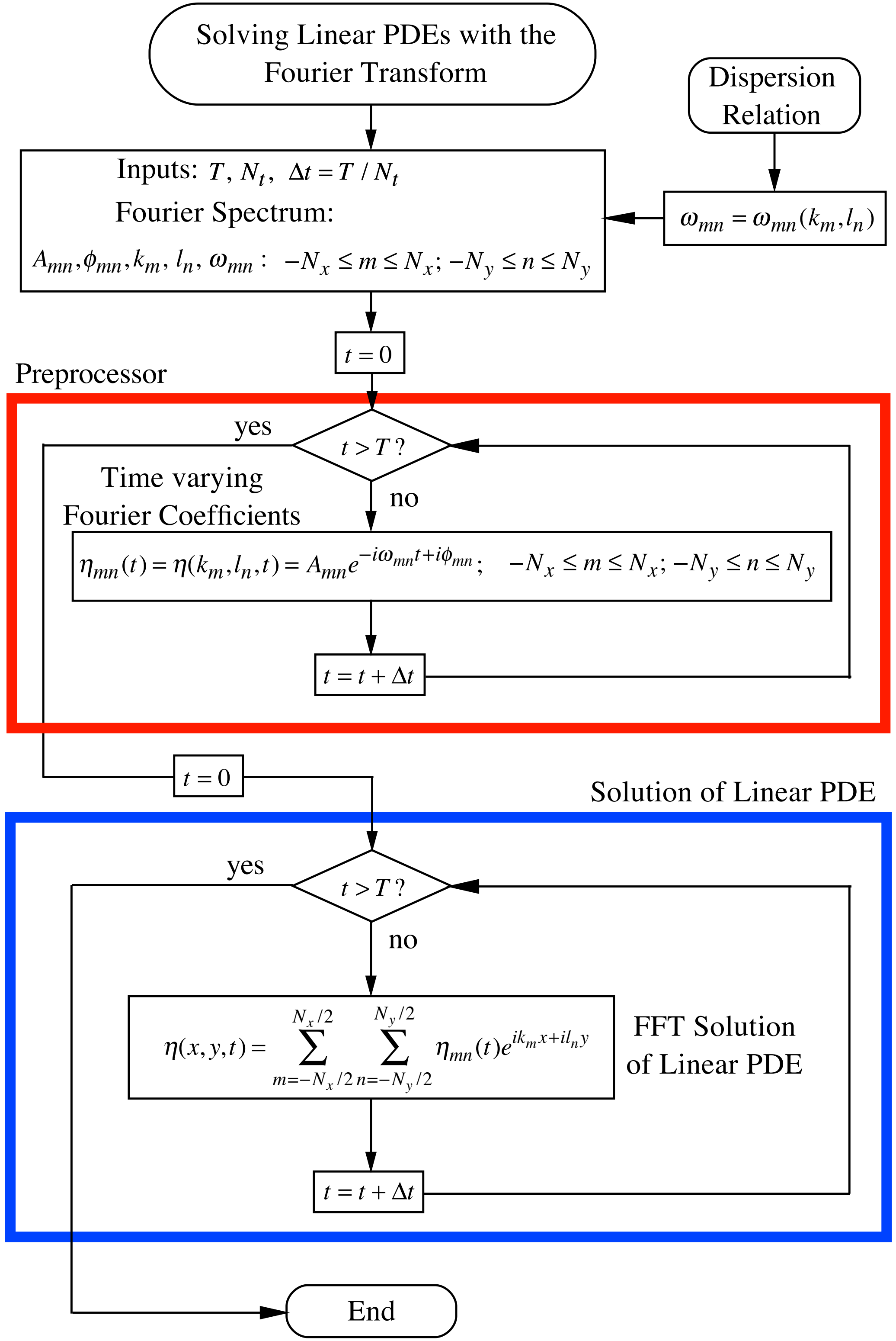
$$u_n(t) = u_n(0) e^{-i\omega t}$$

Dispersion Relation

$$\omega = \omega(k) = c_o k - \beta k^3$$

Cauchy Problem:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(0) e^{i(k_n x - \omega_n t)}$$



Solving Linear PDEs with the Fourier Transform
Dispersion Relation
Inputs: $T, N_t, \Delta t = T / N_t$
Fourier Spectrum:
$A_{mn}, \phi_{mn}, k_m, l_n, \omega_{mn}: \; -N_x \le m \le N_x; -N_y \le n \le N_y$
$\omega_{mn} = \omega_{mn}(k_m, l_n)$
$t = 0$
Preprocessor
yes
$t > T$?
no
Time varying
Fourier Coefficients
$\eta_{mn}(t) = \eta(k_m, l_n, t) = A_{mn} e^{-i\omega_{mn} t + i\phi_{mn}}; \quad -N_x \le m \le N_x; -N_y \le n \le N_y$
$t = t + \Delta t$
$t = 0$
Solution of Linear PDE
yes
$t > T$?
no
$\eta(x,y,t) = \sum_{m=-N_x/2}^{N_x/2} \sum_{n=-N_y/2}^{N_y/2} \eta_{mn}(t) e^{ik_m x + il_n y}$
FFT Solution
of Linear PDE
$t = t + \Delta t$
End

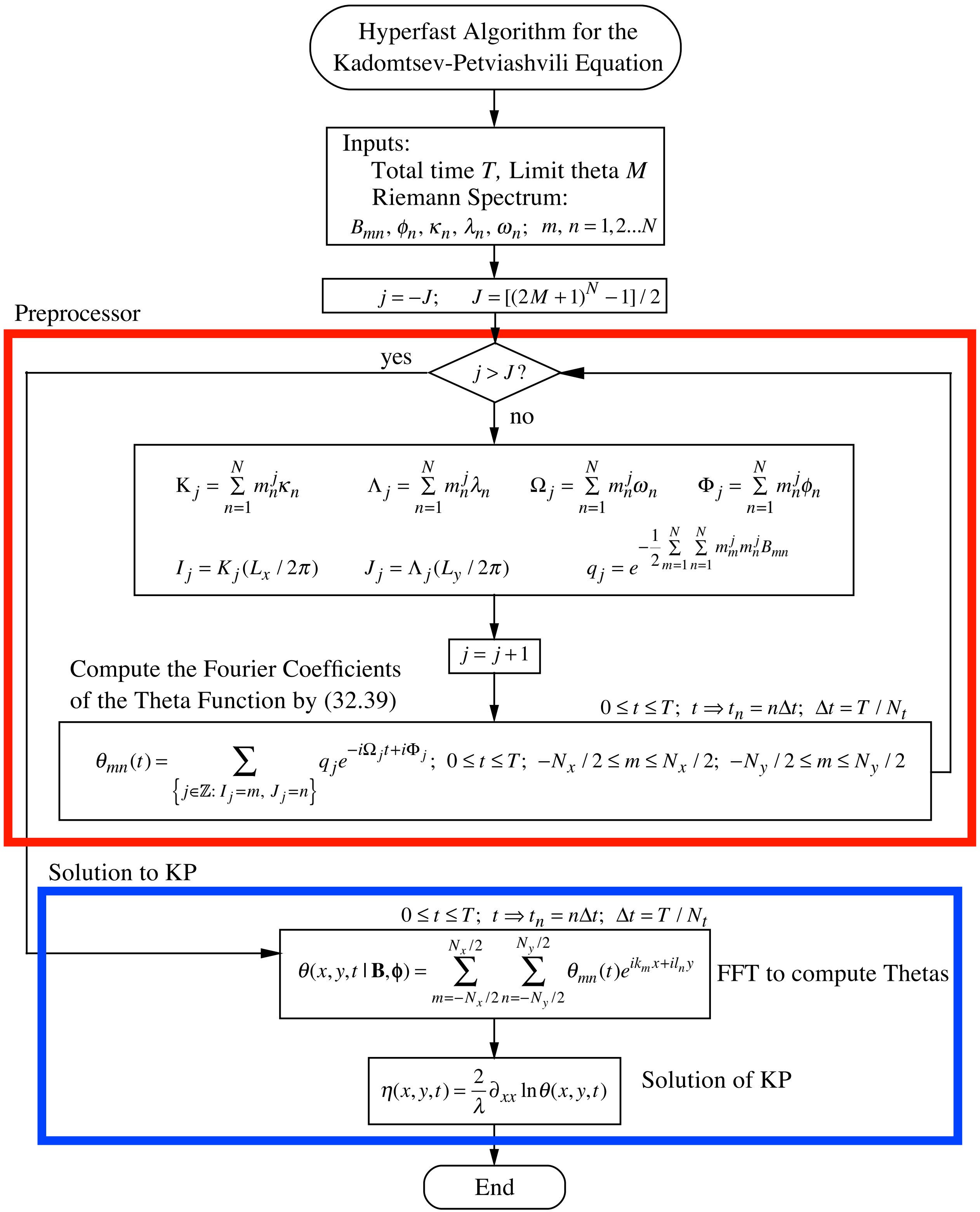
Kadomtsev and Petviashvili (KP) equation

Kadomtsev and Petviashvili (KP) equation

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \gamma\,\partial_x^{-1}\eta_{yy} = 0$$

$$\eta(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t)$$

$$\theta(x,y,t) = \sum_{\mathbf{m}\in\mathbf{Z}} e^{-\frac{1}{2}\mathbf{m}\bullet\mathbf{B}\mathbf{m}+i\mathbf{m}\bullet\mathbf{k}x+i\mathbf{m}\bullet\mathbf{l}y-i\mathbf{m}\bullet\boldsymbol{\omega}t+\mathbf{m}\bullet\boldsymbol{\phi}}$$



Hyperfast Algorithm for the Kadomtsev-Petviashvili Equation
Inputs:
Total time T, Limit theta M
Riemann Spectrum:
$B_{mn}, \phi_n, \kappa_n, \lambda_n, \omega_n; \quad m, n = 1,2...N$
$j = -J; \quad J = [(2M+1)^N - 1]/2$
Preprocessor
yes
$j > J$?
no
$K_j = \sum_{n=1}^{N} m_n^j \kappa_n \quad \Lambda_j = \sum_{n=1}^{N} m_n^j \lambda_n \quad \Omega_j = \sum_{n=1}^{N} m_n^j \omega_n \quad \Phi_j = \sum_{n=1}^{N} m_n^j \phi_n$
$I_j = K_j(L_x/2\pi) \quad J_j = \Lambda_j(L_y/2\pi) \quad q_j = e^{-\frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} m_m^j m_n^j B_{mn}}$
$j = j+1$
Compute the Fourier Coefficients of the Theta Function by (32.39)
$0 \le t \le T; \; t \Rightarrow t_n = n\Delta t; \; \Delta t = T/N_t$
$\theta_{mn}(t) = \sum_{\{j \in \mathbb{Z}: I_j = m, J_j = n\}} q_j e^{-i\Omega_j t + i\Phi_j}; \; 0 \le t \le T; \; -N_x/2 \le m \le N_x/2; \; -N_y/2 \le m \le N_y/2$
Solution to KP
$0 \le t \le T; \; t \Rightarrow t_n = n\Delta t; \; \Delta t = T/N_t$
$\theta(x,y,t \mid \mathbf{B}, \phi) = \sum_{m=-N_x/2}^{N_x/2} \sum_{n=-N_y/2}^{N_y/2} \theta_{mn}(t) e^{ik_m x + il_n y}$
FFT to compute Thetas
$\eta(x,y,t) = \frac{2}{\lambda} \partial_{xx} \ln \theta(x,y,t)$
Solution of KP
End

Partial Theta Summations

$$\theta(x,y,t) = \sum_{m_1=-M}^{M} \sum_{m_2=-M}^{M} \ldots \sum_{m_N=-M}^{M} e^{-\frac{1}{2}\mathbf{m}\cdot\mathbf{B}\mathbf{m}+i\mathbf{m}\cdot\mathbf{k}x+i\mathbf{m}\cdot\mathbf{l}y-i\mathbf{m}\cdot\boldsymbol{\omega}t+\mathbf{m}\cdot\boldsymbol{\phi}}$$

$$\mathbf{m} = [m_1, m_2 \ldots m_N]$$

Exponential

$$(2M+1)^N \sim 10^N$$

Theorem: The Riemann theta function can be reduced to the linear Fourier transform with time dependent coefficients.

$$\theta(x,y,t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \theta_{mn}(t) e^{ik_m x + il_n y}$$

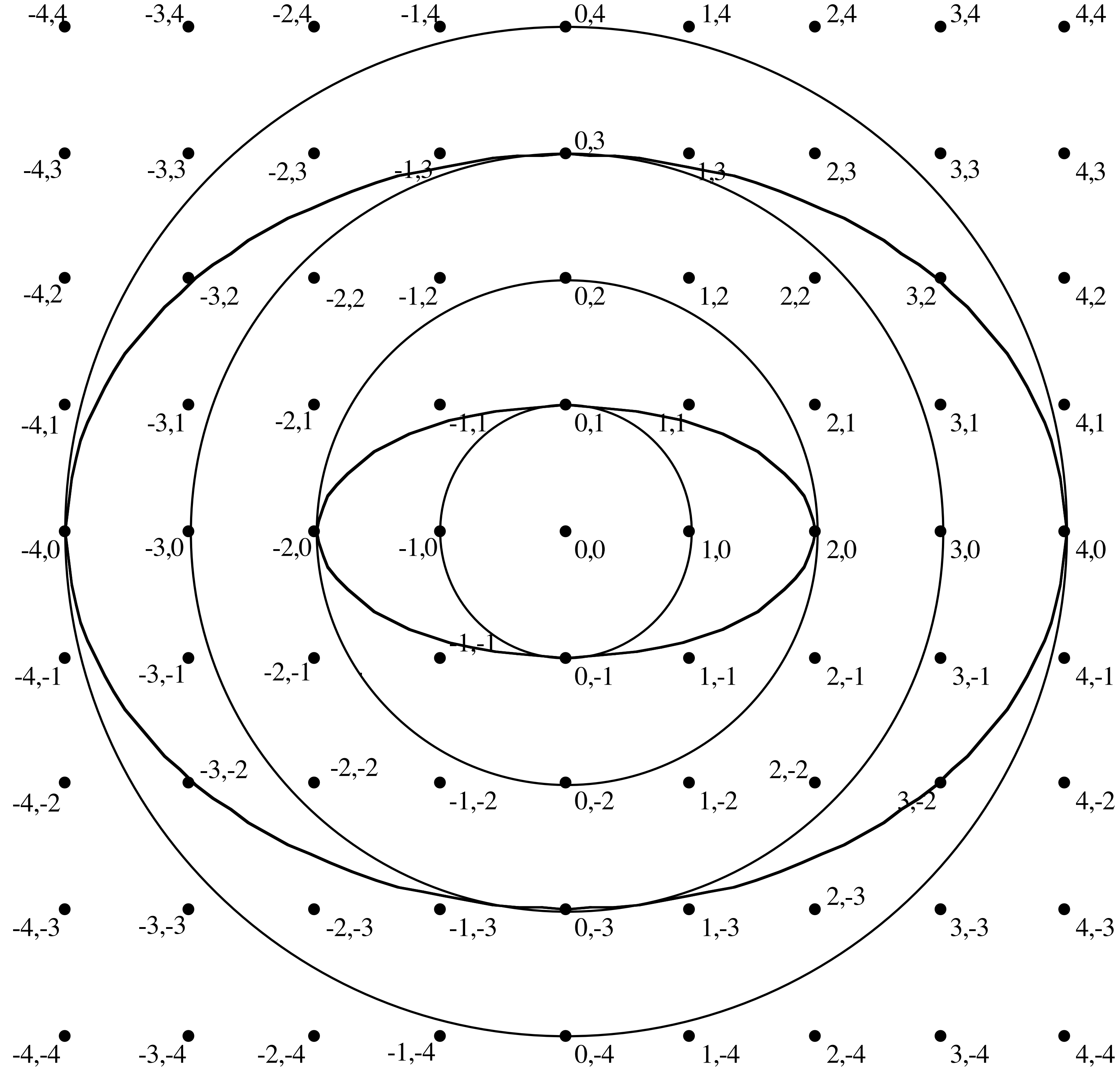
$$\theta_{mn}(t) = \sum_{\substack{\{\mathbf{m}\in\mathbf{Z}:\ L_x(\mathbf{m}\cdot\mathbf{k})/2\pi = m\} \\ \{\mathbf{m}\in\mathbf{Z}:\ L_y(\mathbf{m}\cdot\mathbf{l})/2\pi = n\}}} Q_{\mathbf{m}} e^{-i\mathbf{m}\cdot\boldsymbol{\omega}t}; \qquad Q_{\mathbf{m}} = e^{-\frac{1}{2}\mathbf{m}\cdot\mathbf{B}\mathbf{m} + i\mathbf{m}\cdot\boldsymbol{\phi}}$$

$$\frac{L}{2\pi}\mathbf{m}\cdot\mathbf{k} = \mathbf{m}\cdot\mathbf{n}; \quad \mathbf{n} = [1,2,3\ldots N]; \quad \mathbf{k} = \left[k_1, k_2 \ldots k_N\right]; \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}$$

The ***linear Fourier Coefficients are Fourier series in time***!

Linear Fourier instead

$$\theta_{mn}(t) = \theta_{mn}(0) e^{-i\omega_{mn} t}$$



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-3,-3
-2,-3
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2,-3
3,-3
4,-3
-4,-4
-3,-4
-2,-4
-1,-4
0,-4
1,-4
2,-4
3,-4
4,-4

Summing Over the n-Sphere

N-Cube

$$\theta(x,t\,|\,\mathbf{B},\phi)=\sum_{m_1=-M}^{M}\sum_{m_2=-M}^{M}\cdots\sum_{m_N=-M}^{M} e^{-\frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}m_m m_n B_{mn}}\; e^{i\sum_{n=1}^{N}m_n\kappa_n x-i\sum_{n=1}^{N}m_n\omega_n t+i\sum_{n=1}^{N}m_n\phi_n}$$

$m_1^2=R^2$	line (1D)	$\Rightarrow m_1=R=2M$
$m_1^2+m_2^2=R^2$	circle (2D)	$\Rightarrow m_2=\sqrt{R^2-m_1^2}$
$m_1^2+m_2^2+m_3^2=R^2$	sphere (3D)	$\Rightarrow m_3=\sqrt{R^2-m_1^2-m_2^2}$
$m_1^2+m_2^2+m_3^2+\ldots+m_N^2=R^2$	n-sphere (nD)	$\Rightarrow m_N=\sqrt{R^2-m_1^2-m_2^2-m_3^2-\ldots}$

N-Sphere

$$\theta(x,t)=\sum_{m_1=-M}^{M}\;\sum_{m_2=-I\sqrt{M^2-m_1^2}}^{I\sqrt{M^2-m_1^2}}\;\sum_{m_3=-I\sqrt{M^2-m_1^2-m_2^2}}^{I\sqrt{M^2-m_1^2-m_2^2}}\cdots\sum_{m_N=-I\sqrt{M^2-m_1^2-m_2^2\ldots-m_{N-1}^2}}^{I\sqrt{M^2-m_1^2-m_2^2\ldots-m_{N-1}^2}}\times$$

$$e^{-\frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}m_m m_n B_{mn}}\; e^{i\sum_{n=1}^{N}m_n\kappa_n x-i\sum_{n=1}^{N}m_n\omega_n t+i\sum_{n=1}^{N}m_n\phi_n}$$

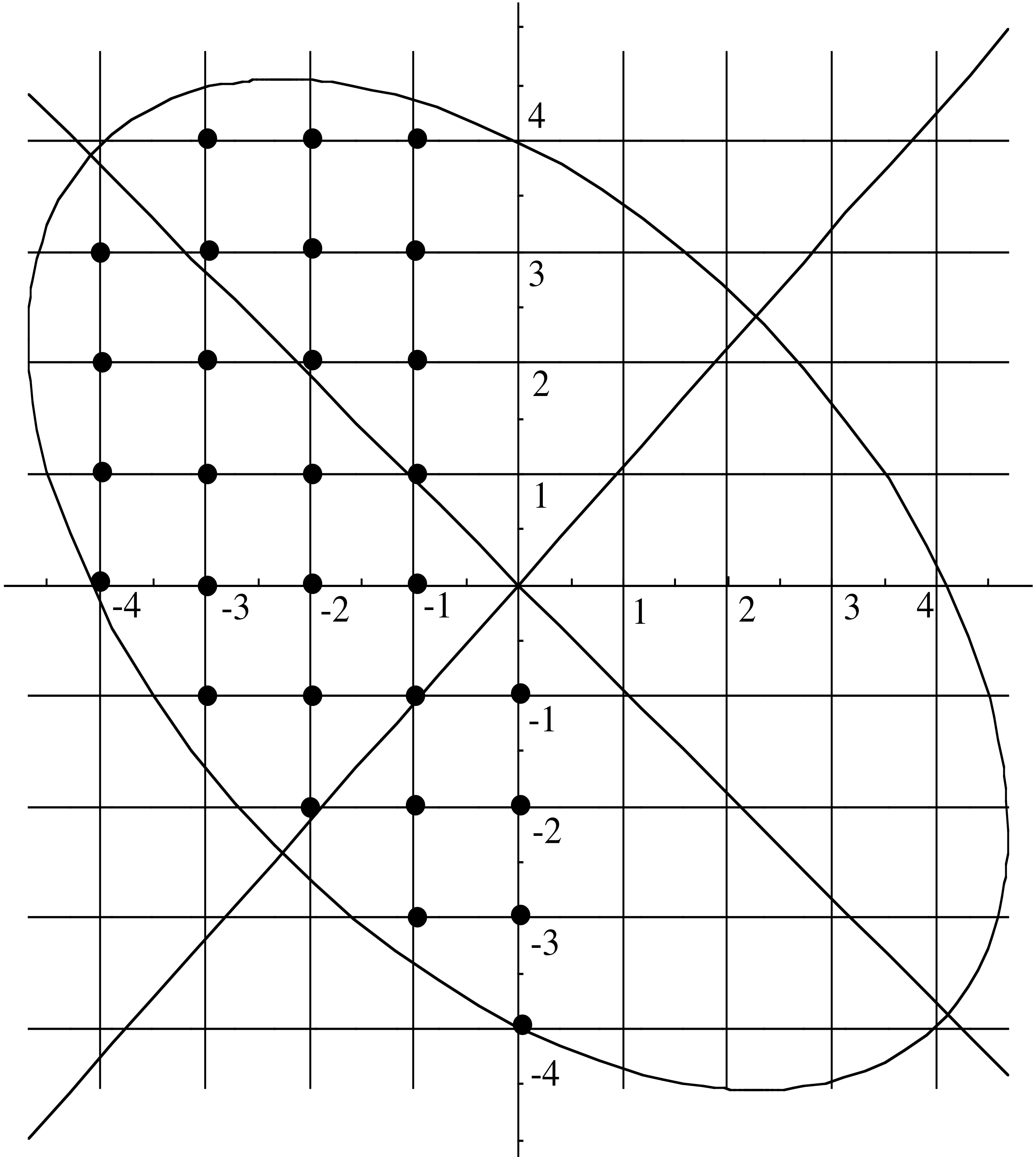
Ratio of Volumns of n-Sphere to n-Box
1
0.1
0.01
0.001
0.0001
0.00001
1x10-6
1x10-7
1x10-8
1x10-9
1x10-10
1x10-11
1x10-12
1x10-13
1x10-14
1x10-15
1x10-16
1x10-17
1x10-18
1x10-19
1x10-20
1x10-21
1x10-22
1x10-23
1x10-24
1x10-25
1x10-26
1x10-27
1x10-28
1x10-29
1x10-30
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51
n

Summing Thetas Over the N-Ellipsoid

$$B_{nn}M_n^2 + 2\left(\sum_{j=1}^{n-1} m_j B_{jn}\right)M_n + \sum_{j=1}^{n-1}\sum_{k=1}^{n-1} m_j m_k B_{jk} + 2\ln q_o = 0$$

$$M_n^{\pm} = -\frac{\sum_{j=1}^{n-1} m_j B_{jn}}{B_{nn}} \pm \sqrt{R_{nn}^2 - \frac{\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} m_j m_k B_{jk}}{B_{nn}} + \left(\frac{\sum_{j=1}^{n-1} m_j B_{jn}}{B_{nn}}\right)^2} \;; \qquad R_{nn}^2 = \frac{2\left|\ln q_o\right|}{B_{nn}}$$

$$\theta(x,t) = \sum_{m_1=-M_1^-}^{M_1^+} \sum_{m_2=-M_2^-}^{M_2^+} \cdots \sum_{m_N=-M_N^-}^{M_N^+} q_{m_1 m_2 \ldots m_N} e^{i\sum_{j=1}^{N} m_j k_j x - i\sum_{j=1}^{N} m_j \omega_j t + i\sum_{j=1}^{N} m_j \phi_j}$$



10–16

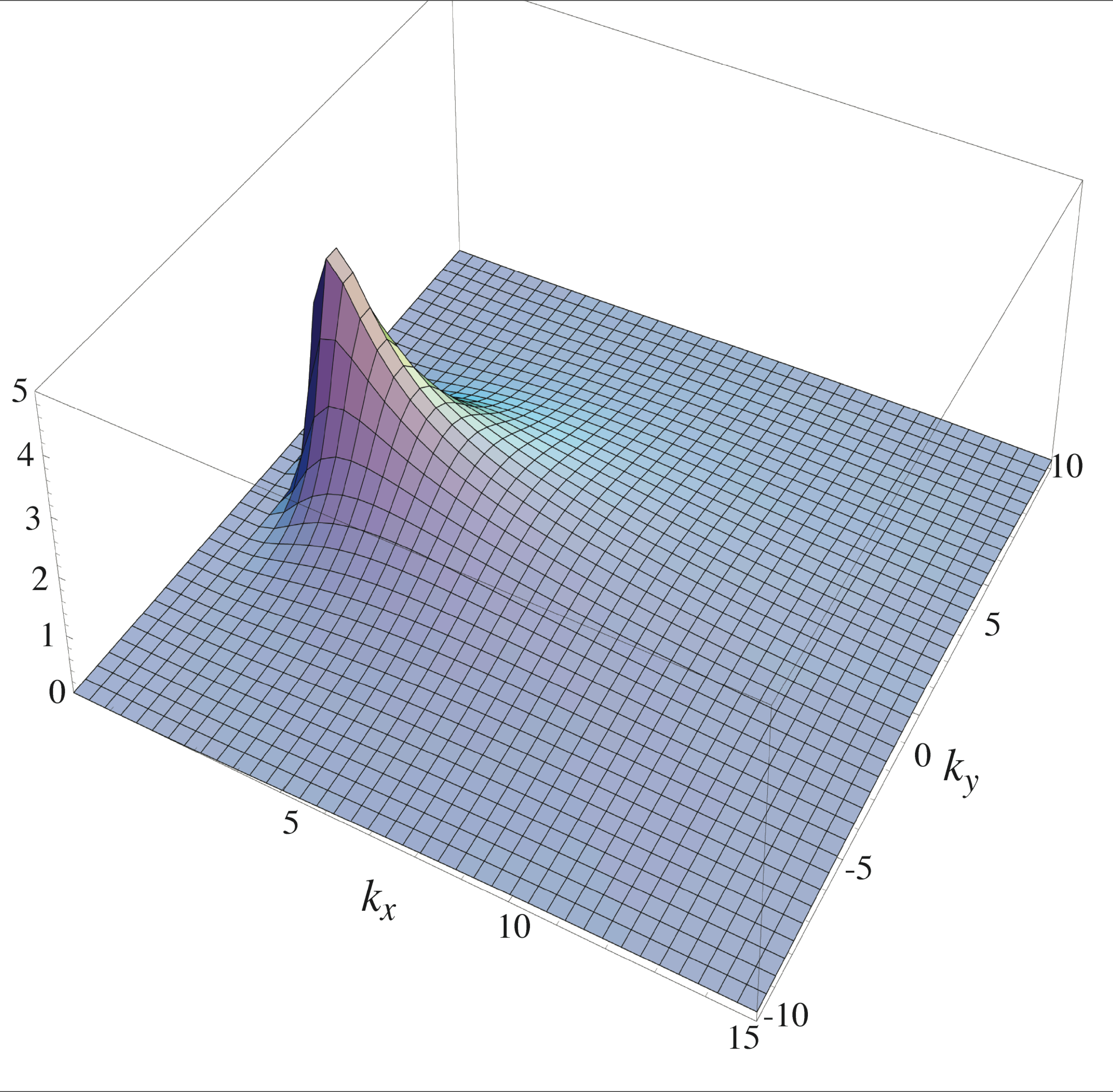
Nonlinear Transformation

$$\eta(x,y,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t); \qquad \lambda = \alpha / 6\beta$$

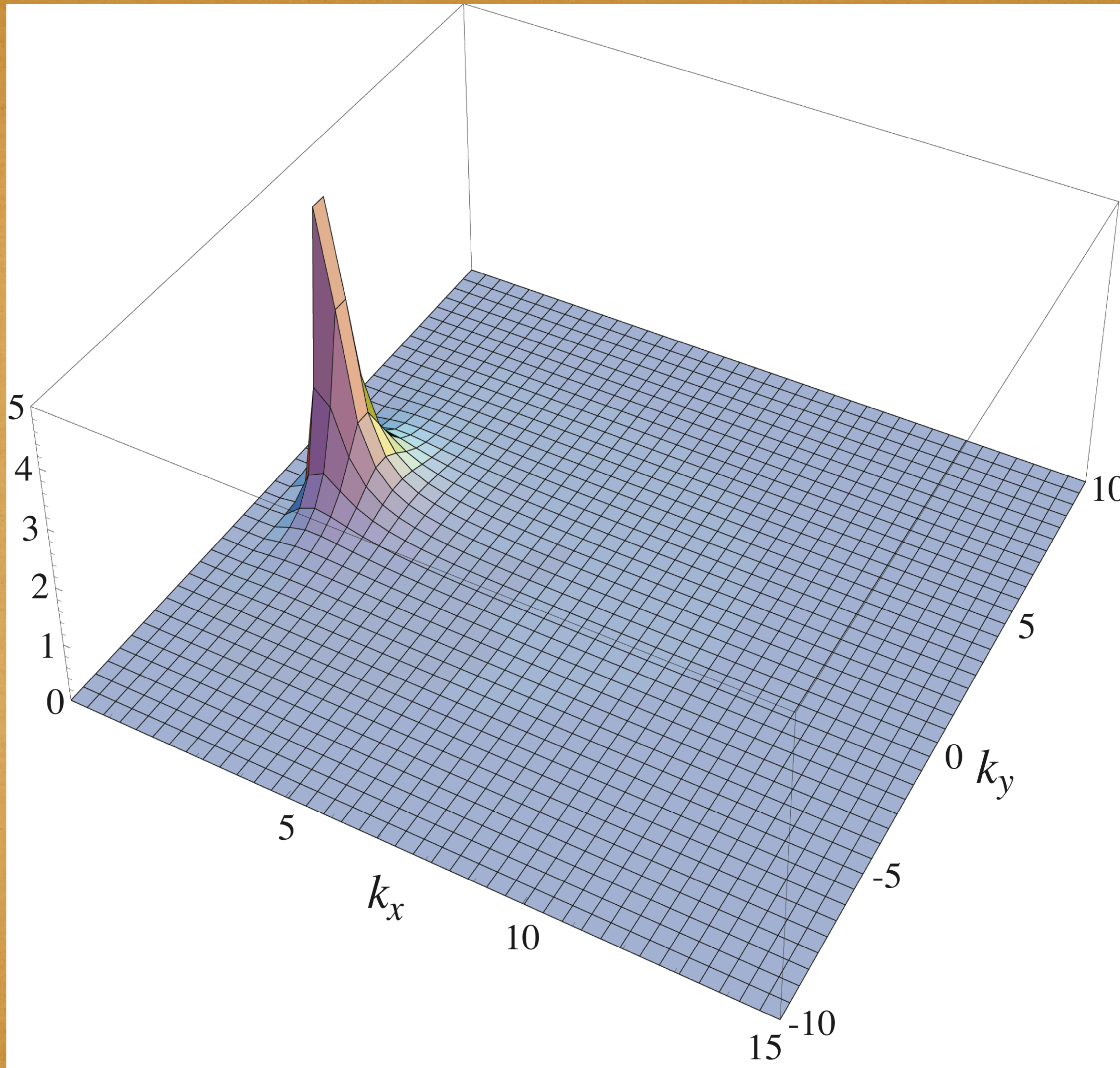
$$\theta(x,y,t) = \exp\left[\frac{\lambda}{2}\iint_x \eta(x,y,t)\,dx'dx''\right]$$

The theta function has a much more narrow spectrum than the surface elevation!

Surface Elevation spectrum



Theta Function spectrum



Euler Equations