FOUR -WAVE RESONANT INTERACTIONS IN THE SHALLOW WATER BOUSSINESQ EQUATIONS

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I. INTRODUCTION

A part of the wave community shares the common thinking that, while the four-wave (*resonant*) interactions are relevant in deep water, the three-wave interactions (*never resonant*, except in the dispersiveless case) become the main nonlinear mechanism of energy transfer as the waves travel into shallow areas. As a consequence, it appears that, when waves travel in shallow water, the four-wave kinetic equation (the Hasselmann equation based on the four-wave resonant interactions) should not be the appropriate tool for wave forecasting and it should be substituted by different models based on i) the computation of the non-resonant triplets, ii) the deterministic modeling (Boussinesq equations or higher order) and iii) the evolution equations for spectra and bispectra; see the recent review [1]. Each of these methods furnishes reasonable results when compared to experimental data but, apart from being time consuming and not always suitable for operational forecasting, none of them covers the lack of knowledge on the physics of waves traveling from deep to shallow water. As a matter of fact, it appears from most of the existing literature that, from deep to shallow water, there is a sort of 'phase transition' and a different physics should be used when wave approach the shallow regions.

In this paper our aim is to show that such a transition is not so drastic; we will show that a correct computation of the four-wave resonant interaction in shallow water can be relevant for understanding the energy transfer in shallow water regions. In order to do that we concentrate our analysis on the simplest (and probably the oldest) deterministic shallow water model, i.e., the Boussinesq equations in flat bottom conditions; we will show analytically the following results:
i) Exact four-wave resonant interactions are included in the flat bottom Boussinesq equations;
ii) The typical nonlinear time scale, τ_{nl}, of the four-wave kinetic equation in shallow water is of the order of $(kh)^4/(\omega\varepsilon^4)$, which can be much faster than the time scale of the nonlinear interaction in deep water;
iii) The four-wave kinetic equation derived from the Boussinesq equations is nothing but the limit of the arbitrary depth Hasselmann equation as $kh \to 0$; in oder words, the Hasselmann equation is 'compatible' with the Boussinesq equations.

The derivation of the above results also offers us the possibility to discuss some more technical issues related to the 'integrability' of wave systems in shallow water. We will show that, if the narrow band approximations is taken, the Zakharov equation in shallow water reduces to the shallow water Davey-Stewartson equation. The equation is known to be integrable and, as a consequence, it does not admit a net flux of energy or wave action across the wave spectrum. The results are reported in section IV.

We will also concentrate our analysis on the equilibrium range in wave spectra: in deep water it is well accepted that the inertial range in the wave spectrum is characterized by a power law of the form of ω^{-4}. As the water become shallower it is plausible that such law could change (this is because the coupling coefficient in the multiple integral in the nonlinear source term is a function also of thewater depth). It has been shown by Zakharov [2] that, in the limit of shallow water, the pawer law in the wave spectrum should be of the form $\omega^{-4/3}$. This results, which has been observed experimentally by Smith and Vincent [3], is perfectly consistent with a dimensional analysis of our shallow water four-wave kinetic equation derived from the deterministic Boussinesq equations. Discussions and conclusions are reported in section VII.

We mention that the present manuscript has been inspired by the following references: [2], [4], [5], where first ideas on the comprehension of the four-wave kinetic equation in shallow water has been reported.

II. THE BOUSSINESQ EQUATIONS AND FOUR-WAVE INTERACTIONS

The Boussinesq equations have been a major tool for studying the dynamics of shallow water waves. They have been derived from the Euler equation in the limit of shallow water and small amplitude waves. In particular two small parameters can be introduced: the first one is $\beta = (kh)^{1/2}$ where k corresponds to wave numbers and h is the fluid depth. the second one is $\alpha = a/h$ where a is a typical wave amplitude. For the derivation one needs to make the hypothesis that $\beta \sim \alpha << 1$ and to neglect in the expansion terms of the order of $\alpha\beta$, β^2 and α^2. Under the hypothesis of waves propagating in only one direction the celebrated Korteweg and De Vries equation can be derived [6].

Our starting point are the classical Boussinesq equations as given in text books (see Whitham [6] and Mei [7]):

$$\eta_t + \nabla \cdot [(\eta + h)\overline{\mathbf{u}}] = 0, \tag{1}$$

$$\overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + g\nabla\eta - \frac{h^2}{3}\nabla\nabla \cdot \overline{\mathbf{u}}_t = 0. \tag{2}$$

Here $\overline{\mathbf{u}} = \overline{\mathbf{u}}(x, y, t)$ is the depth integrated horizontal velocity. $\eta = \eta(x, y, t)$ is the surface elevation and g is gravity. Equation (2) can be written for the velocity potential using $\overline{\mathbf{u}} = \nabla\phi$ and the resulting equation reads:

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta - \frac{h^2}{3}\nabla^2\phi_t = 0. \tag{3}$$

Equations (1) and (3) are our starting point for the analysis.

A. The Boussinesq equations in Fourier space

We now introduce the Fourier transform of η as:

$$\eta = \int \hat{\eta}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{4}$$

where $\mathbf{k} = (k_x, k_y)$, $\mathbf{x} = (x, y)$ and $\hat{\eta}_{\mathbf{k}} = \hat{\eta}(\mathbf{k}, t)$. Similar definition holds for the velocity potential. A straightforward introduction of the Fourier transform in (1) and (3) leads to the following equations:

$$\frac{\partial\hat{\eta}_0}{\partial t} - hk_0^2\hat{\phi}_0 = \int (L_{0,1,2}\hat{\phi}_1\hat{\eta}_2 + L_{0,2,1}\hat{\phi}_2\hat{\eta}_1)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_0)d\mathbf{k}_{12} \tag{5}$$

$$\gamma_0 \frac{\partial \hat{\phi}_0}{\partial t} + g\hat{\eta}_0 = \int L_{1,2,0} \hat{\phi}_1 \hat{\phi}_2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_0) d\mathbf{k}_{12} \tag{6}$$

where $L_{0,1,2} = (\mathbf{k}_0 \cdot \mathbf{k}_1)/2$ and

$$\gamma_k = 1 + \frac{k^2 h^2}{3}. \tag{7}$$

B. Linear equations and wave action variable

We now consider the linearized equations (5), (6). It is straightforward to show that the resulting dispersion relation is given by

$$\omega_k = c_0 \frac{k}{(1 + k^2 h^2/3)^{1/2}}, \tag{8}$$

where $c_0 = \sqrt{gh}$. It is interesting to note that if one introduces a wave action variable $a_\mathbf{k} = a(\mathbf{k}, t)$ related to the surface elevation and velocity potential respectively as:

$$\hat{\eta}_\mathbf{k} = \left(\frac{\omega_k}{2g} \right)^{1/2} (a_\mathbf{k} + a^*_{-\mathbf{k}}), \tag{9}$$

$$\hat{\phi}_\mathbf{k} = -i \frac{1}{\gamma_k} \left(\frac{g}{2\omega_k} \right)^{1/2} (a_\mathbf{k} - a^*_{-\mathbf{k}}), \tag{10}$$

the coupled linear equations reduce to a single equation:

$$\frac{\partial a_\mathbf{k}}{\partial t} + i\omega_k a_\mathbf{k} = 0 \tag{11}$$

that corresponds to an infinite number of decoupled linear oscillator.

C. Nonlinear equations

We now turn to the nonlinear problem and use the relations between wave action, surface elevation and velocity potential in the nonlinear case. In order to avoid higher order nonlinear dispersive terms we use the following approximations in the right hand side of equations (9) and (10) to get:

$$\left(\frac{\omega_k}{2g} \right)^{1/2} \simeq \frac{1}{\sqrt{2}} \left(\frac{h}{g} \right)^{1/4} k^{1/2} \tag{12}$$

$$\frac{1}{\gamma_k} \left(\frac{g}{2\omega_k} \right)^{1/2} \simeq \frac{1}{\sqrt{2}} \left(\frac{g}{h} \right)^{1/4} \frac{1}{k^{1/2}} \tag{13}$$

Applying the same procedure as for the linear case we obtain the following equation:

$$\frac{\partial a_0}{\partial t} + i\omega_0 a_0 = -i \int V_{012} [a_1 a_2 \delta_0^{12} + 2a_1^* a_2 \delta_{01}^2 + a_1^* a_2^* \delta_{012}] d\mathbf{k}_{12}, \tag{14}$$

where

$$V_{012} = \frac{1}{4\sqrt{2}}\left(\frac{g}{h}\right)^{1/4} \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)k_0 + (\mathbf{k}_0 \cdot \mathbf{k}_2)k_1 + (\mathbf{k}_0 \cdot \mathbf{k}_1)k_2}{(k_0 k_1 k_2)^{1/2}} \tag{15}$$

Similar equation has also been found by Zakharov starting from the arbitrary depth Euler equation in the limit of shallow water [2].

III. EFFECTIVE NONLINEAR ENERGY TRANSFER

A. Quasi-resonant three wave interactions

In the case of non dispersive waves, i.e. $\omega(k) = \sqrt{gh}k$, three wave interactions are resonant. Nevertheless, there is a case of major relevance for which, even in the dispersive case, the interaction becomes asymptotically resonant: consider two waves $\mathbf{k}_1$ and $\mathbf{k}_2$ very close to each other and almost parallel, say along the x direction, then the second integral term in equation (14) produce a very long wave number $k_0 = \Delta k = k_{x2} - k_{x1}$. We now calculate the corresponding resonance condition, i.e. $\omega_0 + \omega_1 - \omega_2 = -\frac{1}{2}\sqrt{gh}h^2 k_{x2} k_{x1} \Delta k \simeq -\frac{1}{2}\sqrt{gh}(hk_{x1})^2 \Delta k$. In shallow water $(hk_{x1})^2$ is a small number which is multiplied by Δk which is also small if $k_{x2} \simeq k_{x1}$. So if $\Delta k/k_{x1} \sim (kh)^2 \sim a/h$, the frequency mismatch is of higher order with respect the Boussinesq approximation, and the three wave interactions are to be considered resonant. The result is that, from a practical point of view, the three wave interaction bring to an irreversible transfer of energy.

B. Removing 3 wave non resonant interaction with multiple scale expansion

Apart from the aforementioned case, three wave interactions are never resonant; therefore, it is possible to remove quadratic nonlinearity from equation (14) . There are different way to do it. If the system is Hamiltonian it is natural to use the so called canonical transformation, i.e. a mapping to a different variable which evolve only on the time scale of four-wave interactions which may be resonant; this methodology is explained in details in [8]. The other possibility consists in using the so called multiple scale expansion; this method does not require that the system is Hamiltonian and has been used in [9] to derive originally the Zakharov equation. The resulting equation are not necessary Hamiltonian. Note however that in the multiple scale method one has also some degrees of freedom which allows one to get the final coupling coefficients with the desired symmetries. The equation (14) is in Hamiltonian form therefore any method can be used. Here we have used the method of the multiple scales because it can be used also in the cases where the primitive equations do not have an Hamiltonian structure and therefore can be for example extended to the case of variable depth. The idea under the method of the multiple scales is to let variable $a_{\mathbf{k}}(t)$ be a function of slower time scales, i.e. $a_{\mathbf{k}}(t,\tau)$ with $\tau = \epsilon^2 t$ and look for an evolution equation for the the slow amplitude variable.

We look for a solution of equation (14) of the form:

$$a_{\mathbf{k}}(t,\tau) = b_{\mathbf{k}}(t,\tau) + \epsilon s_{\mathbf{k}}(t,\tau) + \epsilon^2 q_{\mathbf{k}}(t,\tau) + ... \tag{16}$$

Inserting this solution in equation (14) and recalling that $\partial/\partial t = \partial/\partial t + \epsilon^2 \partial/\partial\tau$, we get the equivalent of the Zakharov equation for the evolution of free wave in the shallow water limit:

$$\frac{\partial b_0}{\partial t} + i\omega_0 b_0 = -i\int T_{0123} b_1^* b_2 b_3 \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)] d\mathbf{k}_{123} \tag{17}$$

where $T_{1,2,3,4}$ has the following form:

$$
\begin{aligned}
T_{1,2,3,4} = & - V_{1,3,1-3}V_{4,2,4-2}\left[\frac{1}{\omega_3+\omega_{1-3}-\omega_1}+\frac{1}{\omega_2+\omega_{4-2}-\omega_4}\right] \\
& - V_{2,3,2-3}V_{4,1,4-1}\left[\frac{1}{\omega_3+\omega_{2-3}-\omega_2}+\frac{1}{\omega_1+\omega_{4-1}-\omega_4}\right] \\
& - V_{1,4,1-4}V_{3,2,3-2}\left[\frac{1}{\omega_4+\omega_{1-4}-\omega_1}+\frac{1}{\omega_2+\omega_{3-2}-\omega_3}\right] \\
& - V_{2,4,2-4}V_{3,1,3-1}\left[\frac{1}{\omega_4+\omega_{2-4}-\omega_2}+\frac{1}{\omega_1+\omega_{3-1}-\omega_3}\right] \\
& - V_{1+2,1,2}V_{3+4,3,4}\left[\frac{1}{\omega_{1+2}-\omega_1-\omega_2}+\frac{1}{\omega_{3+4}-\omega_3-\omega_4}\right] \\
& - V_{-1-2,1,2}V_{-3-4,3,4}\left[\frac{1}{\omega_{1+2}+\omega_1+\omega_2}+\frac{1}{\omega_{3+4}+\omega_3+\omega_4}\right].
\end{aligned}
\tag{18}
$$

The coupling coefficient is the result of 6 contributions. The first four will be called $quasi-singular$; this is because as will be shown in the next section, in the narrow band approximation some zero appears in the denominator and some care should be considered when such limit is taken. The last two terms do not show any small denominator and will be called *regular* terms.

IV. ON SOME LIMITS OF THE ZAKHAROV EQUATION IN SHALLOW WATER

Here we will consider the narrow-band approximation of the Zakharov equation in shallow water for one and two dimensional propagation. This will help us in understanding the properties of integrability and the effective nonlinear transfer in shallow water.

A. On the long crested, narrow-band approximation: the Nonlinear Schroedinger equation

We now consider the long crested and then narrow-band approximation of the Zakharov equation. In order to write explicitly the equation, the coupling coefficient should be worked out for the case of a single wave, i.e., $T_{0,0,0,0}$. A naive attempt to calculate such a coefficient, taking the narrow band limit of each of the contributions to $T_{0,0,0,0}$ fails because i) the denominators contain zeros (which should properly treated) and ii) for each of the first 4 (out of 6) contributions the narrow band limit depends on the sign of the difference wave vector. For example, if one attempts to calculate the narrow band approximation of the first contribution, it turns out that it depends on the sign of the difference vector $\mathbf{d} = \mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$, which is of no help unless one specifies an ordering of the vectors. Therefore, some caution must be taken in order to evaluate $T_{0,0,0,0}$. In order to properly consider the the narrow band approximation it is necessary to consider the first and fourth contributions together and the second and third (the fifth and sixth contribution do not show any apparent singularity and can be calculated directly). This is due to the fact that if one considers these contributions together, the result is independent of the vector $\mathbf{d}$ but depends on $|\mathbf{d}|$ and the limit of $|\mathbf{d}|$ going to zero can be safely taken. In order to show explicitly the calculation we consider the following contribution:

$$
C_{1,2,3,4} = -V_{1,3,1-3}V_{4,2,4-2}\frac{1}{\omega_3+\omega_{1-3}-\omega_1} - V_{2,4,2-4}V_{3,1,3-1}\frac{1}{\omega_1+\omega_{3-1}-\omega_3}
\tag{19}
$$

which, after using the definition of $V_{1,2,3}$ in equation (15) and considering waves propagating in one direction with $\mathbf{k}_i = (k_0 + \varepsilon_i, 0)$ with k_0 positive and ε_i a small number which, in the narrow band approximation, tends to zero, equation (19) becomes:

$$C_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4} = -\frac{1}{16}\left(\frac{g}{h}\right)^{1/2} k_0^2|\mathbf{d}|\frac{5\omega_{1-3} + 4sign[\mathbf{d}](\omega_1 - \omega_3)}{\omega_{1-3}^2 - (\omega_1 - \omega_3)^2}, \tag{20}$$

wher we still left explicitly $\mathbf{d} = \mathbf{k}_1 - \mathbf{k}_3$. Even though we are in the one dimensional case, we still have used the vector notation because the sign of $\mathbf{d}$ depends on the difference $\varepsilon_1 - \varepsilon_3$, whose sign is not a priori known.

The dispersion relation can be used directly to calculate $\omega_1 - \omega_3$ and ω_{1-3} as a function of $|\mathbf{d}|$:

$$\omega_1 - \omega_3 = \sqrt{gh}\left[[(sign[\mathbf{d}]|\mathbf{d}|)\left(1 - \frac{h^2k_0^2}{2}\right) - \frac{h^2}{6}(sign[\mathbf{d}]|\mathbf{d}|)^3\right] \tag{21}$$

and

$$\omega_{1-3} = \sqrt{gh}(|\mathbf{d}| - \frac{h^2}{6}|\mathbf{d}|^3). \tag{22}$$

Plugging these relations into (20), the *sign* function disappears from the equation and upon taking he limit of $|\mathbf{d}| \to 0$, we obtain the final result:

$$C_{0,0,0,0} = -\frac{9 - 2h^2k_0^2}{4h^3(4 - h^2k_0^2)} \simeq -\frac{9}{16h^3}, \tag{23}$$

where higher order terms in the dispersive parameter k_0h have been neglected. Similar calculation can be performed to the remaining $quasi-singular$ terms, giving a total contribution to the coupling coefficient of $-9/(4h^3)$. We now consider the following *regular* contribution:

$$-V_{1+2,1,2}V_{3+4,3,4}\left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4}\right] = \frac{9}{8h^3} \tag{24}$$

In the last contribution, no small denominators appear so that a straightforward calculation leads to:

$$-V_{-1-2,1,2}V_{-3-4,3,4}\left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4}\right] = -\frac{k_0^2}{32h(1 - \frac{5}{12}h^2k_0^2)} \simeq -\frac{k_0^2}{32h}. \tag{25}$$

For very shallow water this last contribution is negligible with respect to the other contributions therefore as a final result we obtain:

$$T_{0,0,0,0} = -\frac{9}{8h^3}. \tag{26}$$

Substituting this in the Zakharov equation and Taylor expanding ω around wave number k_0 we get:

$$\omega_k = \omega_0 + C_g\chi + \alpha\chi^2 + ... \tag{27}$$

with $\chi = k - k_0$ and

$$\omega_0 = \sqrt{gh}k_0(1 - k_0^2h^2/6) \quad C_g = \sqrt{gh}(1 - \frac{1}{2}k_0^2h^2) \quad \alpha = -\frac{1}{2}\sqrt{gh}h^2k_0 \tag{28}$$

Now, defining

$$\psi_k = \left(\frac{2c_0 k_0}{g}\right)^{1/2} b_k e^{i\omega_0 t} \tag{29}$$

and writing the resulting Zakharov equation in physical space, we obtain the NLS equation in shallow water:

$$\frac{\partial\psi}{\partial t} + C_g \frac{\partial\psi}{\partial x} - i\alpha\frac{\partial^2\psi}{\partial x^2} - i\sigma|\psi|^2\psi = 0, \tag{30}$$

with

$$\sigma = \frac{9}{16}\frac{c_0}{h^4 k_0}. \tag{31}$$

This result is identical to the one obtained by taking the shallow water limit of the arbitrary-depth Nonlinear Schroedinger equation [7],[10]. The resulting NLS equation is known to be integrable.

B. The weakly two dimensional case: the Davey-Stewartson equation

We now let the wave train to have a weakly directional spreading, i.e. we allow for a weak perturbation in the y direction with $(k_y/k_x)^2 << 1$ and of the same order as the shallow water parameter $(kh)^2$. The procedure for calculating the coupling coefficient is similar to the one described in the previous section with the only complication that wave vectors now are written in the following way $\mathbf{k}_i = (k_0 + \chi_i, k_0 + \mu_i)$ with χ_i and μ_i small with respect to k_0; therefore, the vector $\mathbf{d}$ has both x and y components. Carrying out the calculation for $C_{k_0+\chi_i,k_0+\mu_i}$ to the leading order approximation we obtain:

$$\begin{aligned} C_{k_0+\chi_i,k_0+\mu_i} &= -\frac{1}{16}\left(\frac{g}{h}\right)^{1/2} k_0^2|\mathbf{d}_x| \frac{5\omega_{1-3} + 4sign[\mathbf{d}_x](\omega_1-\omega_3)_x}{(\omega_{1-3}^2 - (\omega_1-\omega_3)^2)_x + c_0(\omega_{1-3})_x \frac{(\mu_1-\mu_3)^2}{|\chi_1-\chi_3|}} = \\ &= -\frac{9k_0^2}{16h}\left[\frac{1}{(k_0h)^2 + \frac{(\mu_1-\mu_3)^2}{(\chi_1-\chi_3)^2}}\right], \end{aligned} \tag{32}$$

where we have used the relations $\chi_1 + \chi_2 = \chi_3 + \chi_4$ and $\mu_1 + \mu_2 = \mu_3 + \mu_4$; the subscript x indicates that these quantities are calculated for the one dimensional case as in the previous section. A similar calculation can be performed on the remaining $quasi-singular$ terms. The first of the regular term takes the following form:

$$-V_{1+2,1,2}V_{3+4,3,4}\left[\frac{1}{\omega_{1+2}-\omega_1-\omega_2} + \frac{1}{\omega_{3+4}-\omega_3-\omega_4}\right] = \frac{9}{8h^3} \tag{33}$$

while the last regular term is of higher order. The final form of the coupling coefficient becomes:

$$T_{k_0+\chi_i,k_0+\mu_i} = \frac{9k_0^2}{8h}\left[\frac{1}{(kh)^2} - \left(\frac{1}{(k_0h)^2 + \frac{(\mu_1-\mu_3)^2}{(\chi_1-\chi_3)^2}}\right) - \left(\frac{1}{(k_0h)^2 + \frac{(\mu_2-\mu_3)^2}{(\chi_2-\chi_3)^2}}\right)\right] \tag{34}$$

The disperion relation then becomes:

$$\omega_k = \omega_0 + C_g\chi + \alpha\chi^2 + \beta\mu^2; \tag{35}$$

with

$$\beta = \frac{c_0}{2k_0} \tag{36}$$

The coupling coefficient in (34) together with the dispersion relation in 35 have a incredibly remarkable property: the coupling coefficient turns out to be exactly zero on the resonant manifold, i.e. in that region of space that satisfies the following relations:

$$\omega_1 + \omega_2 = \omega_3 + \omega_4 \qquad \chi_1 + \chi_2 = \chi_3 + \chi_4 \qquad \mu_1 + \mu_2 = \mu_3 + \mu_4 \tag{37}$$

In order to prove that we eliminate χ_2 and μ_2 from the above expressions and solve the resulting quadratic equation for χ_1 to obtain:

$$\chi_1 = \frac{1}{2}(\chi_3 + \chi_4) \pm \frac{1}{2}\frac{\sqrt{4(\mu_1 - \mu_3)(\mu_1 - \mu_4) + (hk_0)^2(\chi_3 - \chi_4)^2}}{k_0 h} \tag{38}$$

Inserting this relation into the coupling coefficient (34), after some algebraic manipulation, it turns out that the coupling coefficient is exactly zero. This result is related to the fact that the Zakharov equation obtained in the weakly two dimensional case belongs to the family of integrable equations. Indeed, it is straightforward to show that the resulting equation corresponds to the shallow water Davey-Stewartson equation which is known to be integrable. The arbitrary depth Davey Stewartson equation has been derived by Davey and Stewartson in 1974 using the method of the multiple scale, directly applied to the Euler equations. In the shallow water limit this equations has the follow form:

$$\begin{aligned} &\frac{\partial\psi}{\partial t} + C_g\frac{\partial\psi}{\partial x} - i\alpha\frac{\partial^2\psi}{\partial x^2} + i\beta\frac{\partial^2\psi}{\partial y^2} = i\mu|\psi|^2\psi + i\nu\psi\frac{\partial\phi}{\partial x} \\ &(k_0 h)^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \rho\frac{\partial|\psi|^2}{\partial x} \end{aligned} \tag{39}$$

with

$$\rho = -3c_0/(2h^2), \qquad \mu = -\frac{9}{8}\frac{c_0}{h^4 k_0}, \qquad \nu = -\frac{3}{2}k_0. \tag{40}$$

Note that in the case there is no dependence of the y coordinate, the equation reduces to the defocusing Nonlinear Schroedinger equation described in the previous section in equation (30). In order to show that the shallow water weakly two-dimensional Zakharov equation corresponds to the above equation it is sufficient to re-write equation (39) in Fourier space by simply applying the Fourir Transform definition. The equation for the mean flow can be solved directly and the substitute in the equation for the envelope. The resulting coupling coefficient corresponds exactly to the one in (34) provided the relation (29) is used.

V. THE STATISTICAL DESCRIPTION OF THE ZAKHAROV EQUATION IN SHALLOW WATER

Starting from the deterministic Boussinesq equations, in section II we have derived an evolution equation (see equation (17)) on a slower time scale which can be considered as a Zakharov equation in shallow water. It is well known that such an equation is the starting point for the statistical description of water waves. In other words, from equation (17) it is possible to derive an evolution

equation for the wave action spectral density. The procedure is standard and consists in the following approximations:
i) Homogeneity of the wave field: there exists a portion of the ocean where the statistical properties of the surface elevation are space independent; this region should be much larger than the correlation length of the wave field.
ii) The random phase approximation for free waves: the free waves obeying the Zakharov equations should be uncorrelated. This hypothesis is the bases of the closure (just like in turbulence) and allows one to split higher order correlators (which naturally arise once one is interested in writing an evolution equation for the spectrum in a nonlinear wave system) as a combination of the product of lower order correlators.

The resulting evolution equation for the wave action spectral density function is nothing but an Hasselmann-like equation:

$$\frac{\partial N_0}{\partial t} = 4\pi \int |T_{0,1,2,3}|^2 N_0 N_1 N_2 N_3 \left(\frac{1}{N_0}+\frac{1}{N_1}-\frac{1}{N_2}-\frac{1}{N_3}\right)\delta(\mathbf{k}_0+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)]\delta(\omega_0+\omega_1-\omega_2-\omega_3)]d\mathbf{k}_{123} \tag{41}$$

This is the four-wave kinetic equation derived from the deterministic Boussinesq equations which has exactly the same form as the Hasselmann equation in deep water except from the fact that the dispersion relation is different and the coupling coefficient in the multiple integral is also different. In order to give an estimate of the nonlinear time scale of the 4 wave resonant interactions in shallow water, we consider the fact that at the leading non trivial order the coupling coefficient $T_{0,1,2,3}$ scales as k^2/h, therefore we have:

$$\tau_{nl} \sim \frac{\omega}{N^2 T^2 k^4} \sim \frac{1}{\omega}\left(\frac{kh}{\varepsilon}\right)^4 \tag{42}$$

where we have introduced the steepness $\varepsilon = \sigma k$ with σ the standard deviation of the surface elevation. In very shallow water the time scale can be very fast! Note that in deep water the nonlinear time scale is of the order of $1/\omega\varepsilon^4$. If we recall that in the Boussinesq regime $kh^2 \sim a/h$ (or $(kh)^3 \sim \varepsilon$), it appears that he nonlinear time scale can be of the order of:

$$\tau_{nl} \sim \frac{1}{\omega \varepsilon^{8/3}}, \tag{43}$$

that corresponds to more than one order of magnitude faster than the deep water case.

A. Comparison with the arbitrary depth Hasselmann equation

In order to make sure that the kinetic equation derived from the Boussinesq equation is consistent with arbitrary depth Hasselmann equation, a direct comparison of the coupling coefficients is here considered. The coupling coefficient $T^{ad}_{1,2,3,4}$ for arbitrary depth in the Hasselmann equation has the following form:

$$
\begin{aligned}
T^{ad}_{1,2,3,4} = {}& W_{1,2,3,4} \\
& - V^{(-)}_{1,3,1-3} V^{(-)}_{4,2,4-2} \left[\frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4} \right] \\
& - V^{(-)}_{2,3,2-3} V^{(-)}_{4,1,4-1} \left[\frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4} \right] \\
& - V^{(-)}_{1,4,1-4} V^{(-)}_{3,2,3-2} \left[\frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3} \right] \\
& - V^{(-)}_{2,4,2-4} V^{(-)}_{3,1,3-1} \left[\frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3} \right] \\
& - V^{(-)}_{1+2,1,2} V^{(-)}_{3+4,3,4} \left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4} \right] \\
& - V^{(+)}_{-1-2,1,2} V^{(+)}_{-3-4,3,4} \left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4} \right].
\end{aligned} \tag{44}
$$

where the coefficients $V^{(\pm)}_{1,2,3}$ are

$$
V^{(\pm)}_{1,2,3} = \frac{1}{4\sqrt{2}} \left\{ [\mathbf{k_1} \cdot \mathbf{k_2} \pm q_1 q_2] \left(\frac{g\omega_3}{\omega_1\omega_2} \right)^{1/2} + [\mathbf{k_1} \cdot \mathbf{k_3} \pm q_1 q_3] \left(\frac{g\omega_2}{\omega_1\omega_3} \right)^{1/2} + [\mathbf{k_2} \cdot \mathbf{k_3} + q_2 q_3] \left(\frac{g\omega_1}{\omega_2\omega_3} \right)^{1/2} \right\} \tag{45}
$$

with $k_i = |\vec{k}_i|, \omega_i = \omega(k_i)$ and where $q_i = \omega_i^2/g$. $W_{1,2,3,4}$ is given by the following analytical expression:

$$
W_{1,2,3,4} = U_{-1,-2,3,4} + U_{3,4,-1,-2} - U_{3,-2,-1,4} - U_{-1,3,-2,4} - U_{-1,4,3,-2} - U_{4,-2,3,-1} \tag{46}
$$

with

$$
U_{1,2,3,4} = \frac{1}{16} \left(\frac{\omega_3\omega_4}{\omega_1\omega_2} \right)^{1/2} \left[2(k_1^2 q_2 + k_2^2 q_1) - q_1 q_2 \left(q_{1+3} + q_{2+3} + q_{1+4} + q_{2+4} \right) \right]. \tag{47}
$$

The coupling coefficient in the arbitrary depth Hasselmann equation has similar form of the coupling coefficient in the kinetic equation derived from the Boussinesq equations. In order to understand the relation between the Hasselmann equation and the Boussinesq kinetic equation we take the shallow water limit of the coupling coefficients in the Hasselmann equation. If one takes the leading order term in the dispersion relation, i.e. $\omega = \sqrt{ghk}$ and substitute in (45) we obtain that:

$$
V^{(+)}_{1,2,3} = V^{(-)}_{1,2,3} = \frac{1}{4\sqrt{2}} \left\{ \mathbf{k_1} \cdot \mathbf{k_2} \left(\frac{gk_3}{c_0 k_1 k_2} \right)^{1/2} + \mathbf{k_1} \cdot \mathbf{k_3} \left(\frac{gk_2}{c_0 k_1 k_3} \right)^{1/2} + \mathbf{k_2} \cdot \mathbf{k_3} \left(\frac{gk_1}{c_0 k_2 k_3} \right)^{1/2} \right\} \tag{48}
$$

One can recognize that $V^{(+)}_{1,2,3}$ and $V^{(-)}_{1,2,3}$ correspond exactly to the $V_{1,2,3}$ derived from the Boussinesq equations and given in equation (15). It also should be noted that the Hasselmann equation contains an extra term, $W_{1,2,3,4}$, that does not appear in the Boussinesq kinetic equation; nevertheless it is straightforward to show that using $\omega = \sqrt{ghk}$ in such term, then $W_{1,2,3,4} \sim hk^4 + o(h^3k^6)$, therefore is of higher order with respect to the contribution given from V's terms. Our result highlights the fact that there is not a different physics from deep to shallow water. The Hasselmann equation remains still a tool to investigate the nonlinear energy transfer in shallow water.

VI. DIRECT ENERGY CASCADE IN THE BOUSSINESQ EQUATIONS

Here we use a straightforward dimensional analysis of the shallow water kinetic equation in order to determine power law solutions. The Boussinesq equations admit as a constant of motion both the energy and wave action. If these quantities are globally preserved, it means that across some wave number there must be a flux of energy and wave action (we will concentrate only on the energy cascade). In order to determine these power law solutions we make a dimensional analysis of the kinetic equation:

$$\frac{\partial N_{\mathbf{k}}}{\partial t} \sim T^2 N_k^3 k^4 \omega^{-1} \tag{49}$$

For the direct energy cascade, the flux of energy, $\Pi_{\mathbf{k}}$, scales as follows:

$$\Pi_{\mathbf{k}} \sim \omega \frac{\partial N_{\mathbf{k}}}{\partial t} k^2 \sim T^2 N_k^3 k^6. \tag{50}$$

As previously mentioned at leading non trivial order the coupling coefficient scales as $T \sim k^2/h$, and

$$\Pi_{\mathbf{k}} \sim h^{-2} N_k^3 k^{10}, \tag{51}$$

therefore, assuming that there is a window of transparency in k space, i.e., a region of constant flux of energy, the flux must be independent of wave number:

$$N_{\mathbf{k}} \sim k^{-10/3} \qquad or \qquad E_{\mathbf{k}} = \omega_k N_{\mathbf{k}} \sim k^{-7/3} \tag{52}$$

In order to get the isotropic spectrum we still have to consider a multiplication by an extra k to get:

$$E_k \sim k^{-4/3} \sim \omega^{-4/3} \tag{53}$$

This last result is consistent with the finding of Zakharov in [2].

VII. CONCLUSIONS

In this paper we have discussed a number of issues concerning the propagation of waves in shallow water. We have shown that the four-wave kinetic equation derived from the Boussinesq equations is consistent with the arbitrary depth Hasselmann equation. The time scale of the nonlinear transfer in shallow water can be much larger than the one in deep water.

Formally speaking there are some limitations in the use of the four-wave kinetic equation in shallow water which are related to the convergence of the expansion in (16). This limitations have been discussed in [2]. The main problem is related to the fact that in shallow and steep waves the second order term in the expansion is not much smaller than the first order term; therefore, the series cannot be truncated to second or third order. Probably in very shallow water the complete series should be included in order to reconstruct the cnoidal form of the waves. Higher order terms in the expansion would bring higher order resonant wave interactions. The time scale of nonlinear energy transfer is however dominated by the lowest order approximation, therefore our estimate of the nonlinear time scale should still be reasonable, even when the four-wave kinetic equation is not formally applicable.

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